Option-Implied Crash Index

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Joint work with Junxiong Gao from SAIF, Shanghai Jiao Tong University
Introduction

- The seminal work of Black, Scholes, and Merton:

- Models incorporating stochastic volatility and jump risk:
  - Stochastic volatility:
  - Jump-diffusion models:

- Empirical evidence:
The complete-market setting of Black, Merton, and Scholes:
- Model parameter: the diffusion coefficient $\sigma$.
- Application: the option-implied volatility index (VIX).

The jump-diffusion models with the jump component of Merton (1976):
- Model parameter: the mean jump size $\mu$.
- Application: the option-implied crash index (CIX).

The non-parametric approach of Breeden and Litzenberger (1978):
The Stochastic Volatility Model with Jump (SVJ)

The data generating process:

\[ dS_t = \left( r_t - q_t + \eta^s V_t + \lambda V_t (\mu - \mu^*) \right) S_t \, dt + \sqrt{V_t S_t} \, dW_t^{(1)} + dZ_t - \mu S_t \lambda V_t \, dt \]

\[ dV_t = \kappa_v (\bar{V} - V_t) \, dt + \sigma_v \sqrt{V_t} \left( \rho dW_t^{(1)} + \sqrt{1 - \rho^2} \, dW_t^{(2)} \right) \]

The risk-neutral dynamics:

\[ dS_t = (r_t - q_t) S_t \, dt + \sqrt{V_t S_t} \, dW_t^{(Q1)} + dZ_t^Q - \mu^* S_t \lambda V_t \, dt \]

\[ dV_t = \left( \kappa_v (\bar{V} - V_t) + \eta^v V_t \right) dt + \sigma_v \sqrt{V_t} \left( \rho dW_t^{(Q1)} + \sqrt{1 - \rho^2} \, dW_t^{(Q2)} \right) \]

The market prices of risks:

- The diffusion risk premium in index returns: \( \eta^s V_t \).
- The jump risk premium in index returns: \( \lambda V_t (\mu - \mu^*) \).
- The volatility risk premium: \( \eta^v V_t \).
The time-$t$ price of a European-style call option:

$$C_t^{SVJ} = E_t^Q \left[ \exp \left( - \int_t^T r_u \, du \right) (S_T - K)^+ \right] = S_t \, f \left( V_t, \vartheta, r_t, q_t, \tau, \frac{K}{S_t} \right)$$

- $K$ is the strike price and $T = t + \tau$ is the expiration date.
- The latent state variable: $V_t$.
- The model parameters: $\vartheta = (\kappa_v, \bar{v}, \sigma_v, \rho, \lambda, \mu, \sigma_J, \eta^s, \eta^v, \mu^*)$.

Via put/call parity, the time-$t$ price of a European-style put option:

$$P_t^{SVJ} = S_t \, f \left( V_t, \vartheta, r_t, q_t, \tau, \frac{K}{S_t} \right) - S_t \, e^{-q \tau} + K \, e^{-r \tau}.$$
Model Estimation

- We use the joint time-series of the S&P 500 index and options (Pan 2002).
- The moment conditions are constructed on the two state variables:
  - Daily index returns: \( y_t = \ln S_t - \ln S_{t-1} - r_t - q_t \)
  - Time-\( t \) volatility \( V_t \), inferred from an ATM call, given model parameters \( \vartheta \),
    \[
    C_t^{\text{Market}} = C_t^{\text{SVJ}} = S_t f \left( V_t, \vartheta, r_t, q_t, \tau, \frac{K}{S_t} \right).
    \]
- The model parameters \( \vartheta \) and the latent \( V_t \) are simultaneously estimated,
  - using moment conditions based on
    \[
    \begin{align*}
    \varepsilon_{t}^{y1} &= y_t - M_1 (V_{t-1}, \vartheta) \\
    \varepsilon_{t}^{y2} &= y_t^2 - M_2 (V_{t-1}, \vartheta) \\
    \varepsilon_{t}^{y3} &= y_t^3 - M_3 (V_{t-1}, \vartheta) \\
    \varepsilon_{t}^{y4} &= y_t^4 - M_4 (V_{t-1}, \vartheta) \\
    \varepsilon_{t}^{yv} &= y_t V_t - M_7 (V_{t-1}, \vartheta) \\
    \varepsilon_{t}^{v1} &= V_t - M_5 (V_{t-1}, \vartheta) \\
    \varepsilon_{t}^{v2} &= V_t^2 - M_6 (V_{t-1}, \vartheta) \\
    \varepsilon_{t}^{vv} &= y_t V_t - M_7 (V_{t-1}, \vartheta)
    \end{align*}
    \]
  - and pricing errors of 30-day ITM calls and 90-day ATM calls.
## Estimation Results: Model Parameters

### Panel A: SVJ estimation using 1996-2021 daily price data

<table>
<thead>
<tr>
<th></th>
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<tbody>
<tr>
<td>( k_v )</td>
<td>5.88</td>
<td>[45.12]</td>
</tr>
<tr>
<td>( \bar{v} )</td>
<td>0.0108</td>
<td>[8.70]</td>
</tr>
<tr>
<td>( \sigma_v )</td>
<td>0.29</td>
<td>[15.54]</td>
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<tr>
<td>( \rho )</td>
<td>-0.52</td>
<td>[-15.11]</td>
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<tr>
<td>( \lambda )</td>
<td>23.9</td>
<td>[13.71]</td>
</tr>
<tr>
<td>( \mu )</td>
<td>-1.00%</td>
<td>[-0.14]</td>
</tr>
<tr>
<td>( \sigma_J )</td>
<td>2.97%</td>
<td>[5.83]</td>
</tr>
<tr>
<td>( \eta_s )</td>
<td>3.10</td>
<td>[1.51]</td>
</tr>
<tr>
<td>( \eta_v )</td>
<td>3.19</td>
<td>[16.65]</td>
</tr>
<tr>
<td>( \mu^* )</td>
<td>-16.16%</td>
<td>[-37.32]</td>
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</table>

### Panel B: SVJ fitting the joint moments of \( y_t \) and \( V_t \)

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>( y_t )</td>
<td>20.14</td>
<td>[0.26]</td>
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<tr>
<td>( y_t^2 )</td>
<td>0.35</td>
<td>[0.09]</td>
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<tr>
<td>( y_t^3 )</td>
<td>1.15</td>
<td>[0.21]</td>
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<tr>
<td>( y_t^4 )</td>
<td>-0.20</td>
<td>[-0.06]</td>
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<tr>
<td>( V_t )</td>
<td>0.03</td>
<td>[0.04]</td>
</tr>
<tr>
<td>( V_t^2 )</td>
<td>0.01</td>
<td>[0.02]</td>
</tr>
<tr>
<td>( y_t V_t )</td>
<td>1.10</td>
<td>[0.20]</td>
</tr>
</tbody>
</table>
Estimation Results: The Latent State Variable $V_t$

The Model-Implied Variance $V_t$

- $\text{corr}(VIX^2, V) = 98.55\%$
- $\text{corr}(\Delta VIX^2, \Delta V) = 86.39\%$

The Ratio of Model-Implied Variance to $VIX^2$

$\text{avg}(V/VIX^2) = 0.52$
Estimation Results: Model-Implied Jump Arrival Intensity

Average Jump Arrival: 61.91% per year
Estimation Results: Model-Implied Volatility Surface

Varying Jump Arrival Intensity $\lambda V_t$

Varying Mean Jump Size $\mu^*$
Option-Implied Crash Surface

**Vol Surface, Data vs Model**

![Volatility Surface Comparison]

**SVJ Implied Mean Jump Size**

![Mean Jump Size Comparison]

**Option-Implied Crash Surface**
Construct the Option-Implied Crash Index (CIX)

- Infer the SVJ model implied \( \mu^I_t(\tau, K) \) from an OTM put with market price \( P^\text{Market}_t \),
  \[
P^\text{Market}_t = P^\text{SVJ}_t = S_t f \left( V_t, \vartheta^\perp, \mu^I_t(\tau, K), r, q, \tau, \frac{K}{S_t} \right) - S_t e^{-q\tau} + K e^{-r\tau},
\]
  where we fix \( V_t \) and all other parameters \( \vartheta^\perp \) to the estimation results.

- This parallels the Black-Scholes implied \( \sigma^I_t(\tau, K) \),
  \[
P^\text{Market}_t = P^\text{BS}_t = S_t f \left( \sigma^I_t(\tau, K), r, q, \tau, \frac{K}{S_t} \right) - S_t e^{-q\tau} + K e^{-r\tau}.
\]

- Following VIX, we construct CIX by interpolating around the 30-day to expiration,
  \[
  -\text{CIX}_t = \mu^I_t(\tau_1) \frac{\tau_2 - 30}{\tau_2 - \tau_1} + \mu^I_t(\tau_2) \frac{30 - \tau_1}{\tau_2 - \tau_1},
  \]
  using \( \tau_1 \leq 30 < \tau_2 \) and \( \mu^I_t(\tau) \) for each \( \tau \) is averaged across all \( K/S \in [0.93, 0.97] \).
The Crash Index (CIX) vs VIX (right axis)

- corr(CIX, VIX) = 12%
- corr(ΔCIX, ΔVIX) = 6%

Time-Series of CIX vs VIX
Time-Series of CIX vs VIX

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Option-Implied Crash Index
Validating CIX using Non-Parametric Skew

- Since 2003, the CBOE VIX is computed via

\[
e^{-r\tau} E^Q_t \left( R(t, \tau)^2 \right) = \int_{S_t}^{\infty} 2 \left( 1 - \ln \left( \frac{K}{S_t} \right) \right) \frac{C_t(\tau, K)}{K^2} dK + \int_0^{S_t} 2 \left( 1 + \ln \left( \frac{S_t}{K} \right) \right) \frac{P_t(\tau, K)}{K^2} dK.
\]

where \( R(t, \tau) = \ln S_{t+\tau} - \ln S_t \).

- Bakshi, Kapadia, and Madan (2003) computes the risk-neutral skewness via

\[
e^{-r\tau} E^Q_t \left( R(t, \tau)^3 \right) = \int_{S_t}^{\infty} 6 \ln \left( \frac{K}{S_t} \right) - 3 \left( \ln \left( \frac{K}{S_t} \right) \right)^2 \frac{C_t(\tau, K)}{K^2} dK - \int_0^{S_t} 6 \ln \left( \frac{S_t}{K} \right) + 3 \left( \ln \left( \frac{S_t}{K} \right) \right)^2 \frac{P_t(\tau, K)}{K^2} dK.
\]

- Crashes: important in generating negative skewness.
- The non-parametric skewness can be used to validate the information content of our CIX index.
Time-Series of CIX, Skew and IV Spread

The Crash Index (CIX)
Skew = Negative Skewness (right axis)
corr(CIX, Skew) = 65%
corr(ΔCIX, ΔSkew) = 65%

CIX and Skewness

The Crash Index (CIX)
IV Spread (right axis)
corr(CIX, IVsprd) = 68%
corr(ΔCIX, ΔIVsprd) = 53%
corr(Skew, IVsprd) = 89%
corr(ΔSkew, ΔIVsprd) = 67%

CIX and IV Spread
Time-Series of CIX and Skewness

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30
25
20
15
10
5
0

CIX (%)

2008

Jan 2008

Apr 2008

Jul 2008

Oct 2008

Jan 2009

Apr 2009

Jul 2009

2017

Jan

Feb

Mar

Apr

May

Jun

2017

Jan

Feb

Mar

Apr

May

Jun

2020

Jan

Feb

Mar

Apr

May

Jun

2021

Jan

Feb

Mar

Apr

May

Jun
Skewness under the SVJ Model

SVJ Skewness with Varying $\mu^*$

SVJ Skewness with Varying $\rho$
Time-Series of Skewness, Non-Parametric vs the SVJ Model

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Option-Implied Crash Index

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Explaining the Crash Index (CIX)

<table>
<thead>
<tr>
<th>Independent Variable</th>
<th>Coefficient</th>
<th>Standard Error</th>
<th>t-stat</th>
<th>P-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta VIX_t$</td>
<td>6.37</td>
<td>3.30</td>
<td>1.83</td>
<td>0.07</td>
</tr>
<tr>
<td>$\Delta IV_{sprd_t}$</td>
<td>59.42</td>
<td>29.39</td>
<td>2.03</td>
<td>0.04</td>
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<tr>
<td>$\Delta Skew_t$</td>
<td>64.60</td>
<td>31.82</td>
<td>2.03</td>
<td>0.04</td>
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<tr>
<td>$\Delta (P/C)_t$</td>
<td>5.53</td>
<td>3.66</td>
<td>1.52</td>
<td>0.13</td>
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<tr>
<td>$\Delta Noise_t$</td>
<td>2.58</td>
<td>1.66</td>
<td>1.52</td>
<td>0.13</td>
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<tr>
<td>$\Delta Term_t$</td>
<td>-2.70</td>
<td>1.95</td>
<td>-1.41</td>
<td>0.16</td>
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<tr>
<td>$\Delta TED_t$</td>
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<td>1.09</td>
<td>-1.41</td>
<td>0.16</td>
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<tr>
<td>$\Delta Dsprd_t$</td>
<td>1.03</td>
<td>0.68</td>
<td>1.52</td>
<td>0.13</td>
</tr>
<tr>
<td>$CIX_{t-1}$</td>
<td>-0.35</td>
<td>0.34</td>
<td>-0.97</td>
<td>0.34</td>
</tr>
<tr>
<td>$VIX_{t-1}$</td>
<td>0.01</td>
<td>0.01</td>
<td>0.23</td>
<td>0.82</td>
</tr>
<tr>
<td>$Ret_{t-1}$</td>
<td>-2.65</td>
<td>2.67</td>
<td>-0.97</td>
<td>0.34</td>
</tr>
<tr>
<td>$1_{3rd\ Friday}$</td>
<td>1.72</td>
<td>2.38</td>
<td>0.73</td>
<td>0.47</td>
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</table>

Dependent Variable: $\Delta CIX_t$ (%)
## Return Predictability Using Option-Implied Risk Measures

<table>
<thead>
<tr>
<th>Prob(Tail)</th>
<th>CIX</th>
<th>VIX</th>
<th>Skew</th>
<th>IVSprd</th>
<th>Prob(Tail)</th>
<th>CIX</th>
<th>VIX</th>
<th>Skew</th>
<th>IVSprd</th>
</tr>
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<tbody>
<tr>
<td>2%</td>
<td>-47.41</td>
<td>59.88</td>
<td>-29.08</td>
<td>-23.87</td>
<td>2%</td>
<td>20.28</td>
<td>-4.99</td>
<td>25.22</td>
<td>35.70</td>
</tr>
<tr>
<td></td>
<td>[-2.89]</td>
<td>[2.03]</td>
<td>[-2.03]</td>
<td>[-1.47]</td>
<td>5%</td>
<td>19.55</td>
<td>-3.39</td>
<td>21.32</td>
<td>30.17</td>
</tr>
<tr>
<td></td>
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<td></td>
<td>[-2.89]</td>
<td>[2.03]</td>
<td>[-2.03]</td>
<td>[-1.47]</td>
<td>5%</td>
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<td>-3.39</td>
<td>21.32</td>
<td>30.17</td>
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<td>[2.03]</td>
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<td>[-1.47]</td>
<td>5%</td>
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<td>[2.03]</td>
<td>[-2.03]</td>
<td>[-1.47]</td>
<td>5%</td>
<td>19.55</td>
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<td>[2.03]</td>
<td>[-2.03]</td>
<td>[-1.47]</td>
<td>5%</td>
<td>19.55</td>
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<td>21.32</td>
<td>30.17</td>
</tr>
</tbody>
</table>
Return Predictability Using Option-Implied Risk Measures

<table>
<thead>
<tr>
<th>Dependent Variable: Day $t + 1$ SPX Returns (bps)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta CIX_t$</td>
</tr>
<tr>
<td>-24.65 [-4.39]</td>
</tr>
<tr>
<td>$\Delta VIX_t$</td>
</tr>
<tr>
<td>15.14 [2.32]</td>
</tr>
<tr>
<td>$\Delta SKEW_t$</td>
</tr>
<tr>
<td>-19.11 [-3.71]</td>
</tr>
<tr>
<td>$\Delta CIX_t \times \Delta VIX_t$</td>
</tr>
<tr>
<td>-36.98 [-1.51]</td>
</tr>
<tr>
<td>$\Delta CIX_t \times \Delta SKEW_t$</td>
</tr>
<tr>
<td>9.16 [0.68]</td>
</tr>
<tr>
<td>$\Delta CIX_t$</td>
</tr>
<tr>
<td>6.36 [4.76]</td>
</tr>
<tr>
<td>constant</td>
</tr>
<tr>
<td>6.36 [4.76]</td>
</tr>
<tr>
<td>$R^2(%)$</td>
</tr>
<tr>
<td>0.37</td>
</tr>
</tbody>
</table>
Conclusions

- We construct an option-implied Crash Index (CIX) by exploring the pricing difference between the OTM puts and ATM options, benchmarked against the SVJ model.
- The construction of our CIX index is analogous to that of the VIX index:
  - The mean jump size $\mu$ of the SVJ model implied by OTM puts.
  - The volatility parameter $\sigma$ of the Black-Scholes model implied by ATM options.
- Empirically, we find that
  - The CIX index is closely related to the non-parametric option-implied skewness and positively correlated with the put/call volume ratio.
  - Post 2008, the CIX index has increased significantly.
  - Large increases in CIX are followed by large negative SPX returns.
  - By contrast, large increases in VIX are followed by large positive SPX returns.