Options and Black-Scholes Implied Volatility

Financial Markets, Day 2, Class 3

Jun Pan

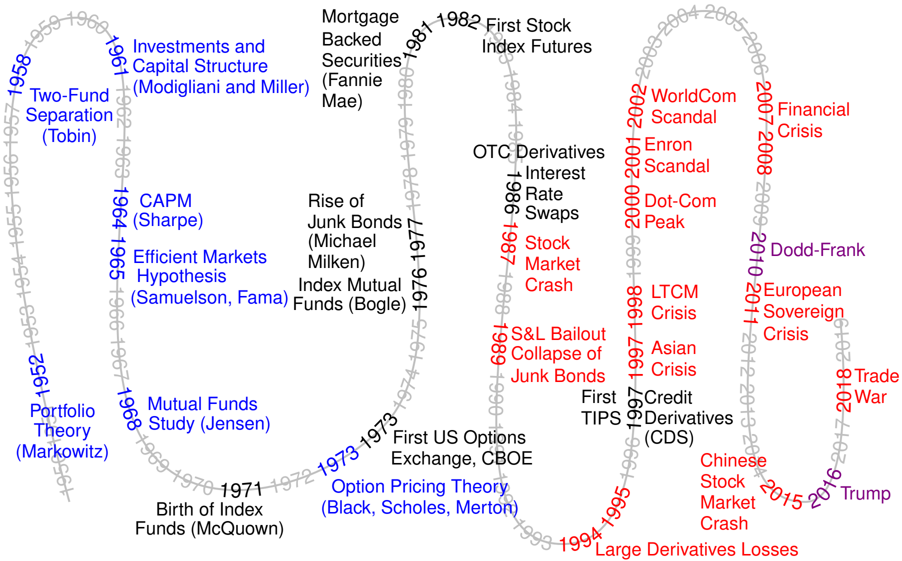
Shanghai Advanced Institute of Finance (SAIF)
Shanghai Jiao Tong University

April 19, 2019

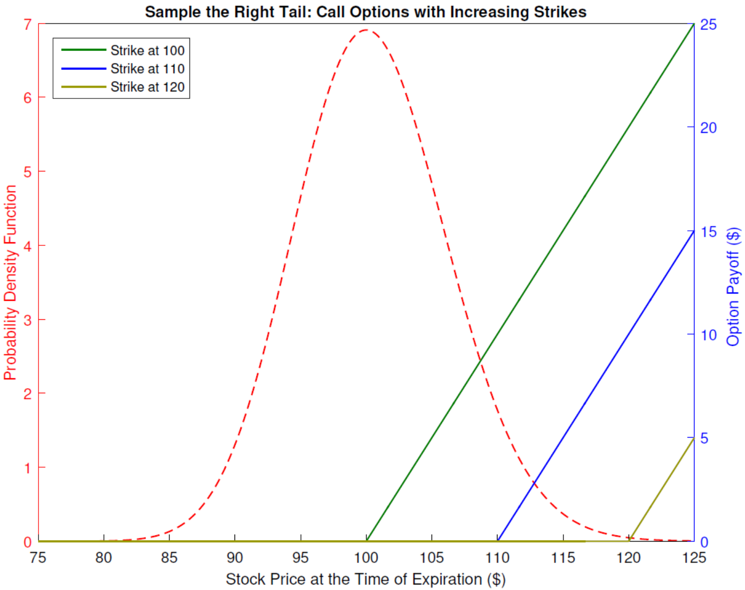
Outline

- An overview of the options market:
  - Why options?
  - History, trading volume and market size.
  - Options exchanges and market participants.
- The Black-Scholes option pricing model:
  - The Black-Scholes model.
  - Risk-neutral pricing.
  - The Black-Scholes formula.
- Using the Black-Scholes formula:
  - The pricing of at-the-money options.
  - Black-Scholes option implied volatility.

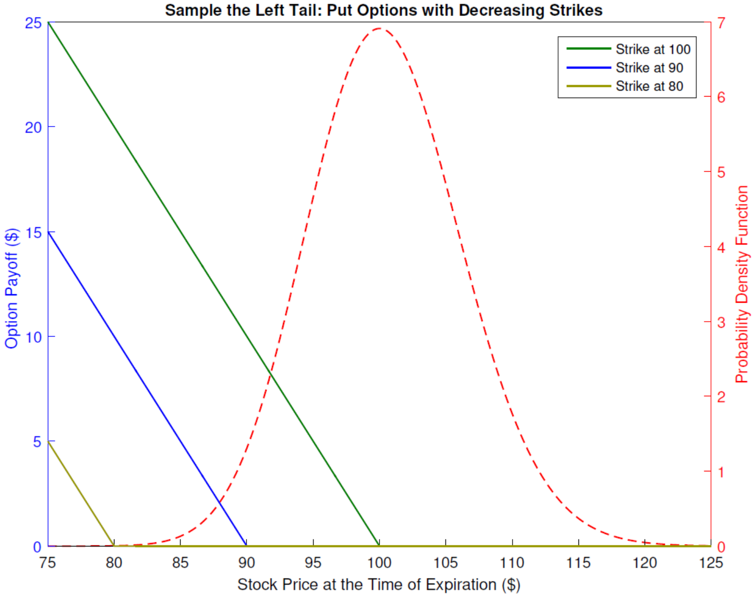
Modern Finance



Sampling the Right Tail



Sampling the Left Tail



A Brief History

- 1973: CBOE founded as the first US options exchange, and 911 contracts were traded on 16 underlying stocks on first day of trading.
- 1975: The Black-Scholes model was adopted for pricing options.
- 1977: Trading in put options begins.
- 1983: On March 11, index option (OEX) trading begins; On July 1, options trading on the S&P 500 index (SPX) was launched.
- 1987: Stock market crash.
- 1993: Introduces CBOE Volatility Index (VIX).
- 2003: ISE (an options exchange founded in 2000) overtook CBOE to become the largest US equity options exchange.
- 2004: CBOE Launches futures on VIX.

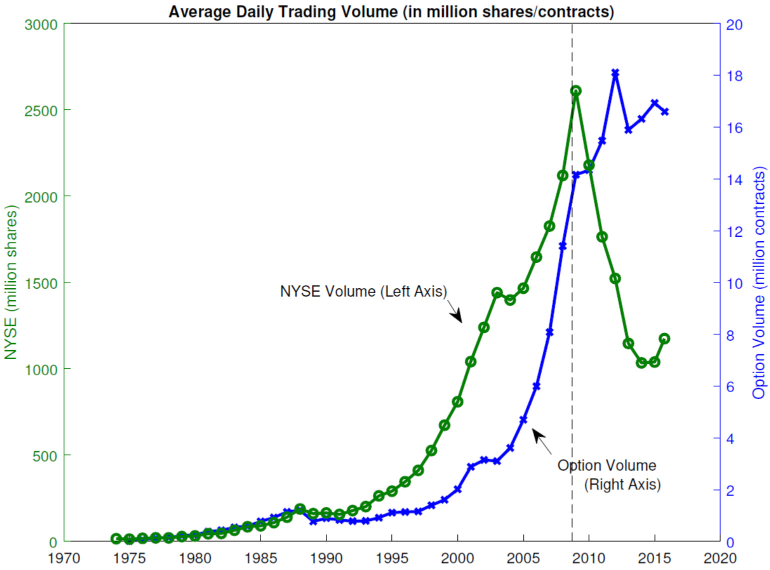
Trading Volumes

OCC Monthly Report for September 2015

	Equity	ETF	Index	SPX	VIX
avg daily contract (million)	7.57	7.33	2.04	1.14	0.76
avg daily premium ($ billion)	1.58	1.50	3.21	2.78	0.14
avg premium per contract ($)	211	205	1,575	2,474	195
put/call ratio of contract	0.80	1.48	1.21	1.87	0.60
put/call ratio of dollar volume	1.14	2.05	1.43	1.48	0.35

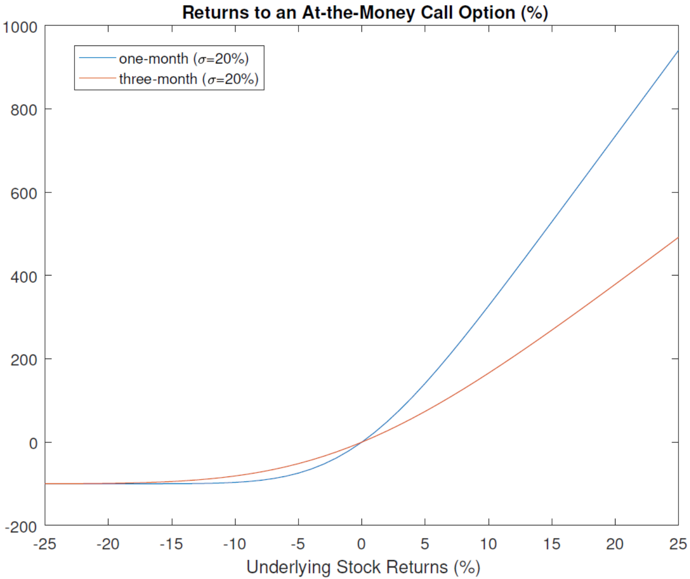
- For Sept 2015, the average daily trading in options is 16.94 million contracts and $6.30 billion; the average daily trading in stocks is 7.92 billion shares and $321 billion.
- End of Sept 2015, the open interest for equity and ETF options is 292 million contracts, and 23.7 million contracts for index options. The overall market size: about $95.7 billion.

Trading Volumes, Stocks vs Options



Source: NYSE and OCC (The Options Clearing Corporation)

Leverage Embedded in Call Options



Leverage Embedded in Put Options

Options Exchanges

- After 6 years of litigation, CBOE in 2012 was able to retain its exclusive licenses on options on the S&P 500 index. As a result, CBOE remains its dominance in index options with over 98% of the market share.
- The trading of equity and ETF options, however, is spread over 12 options exchanges:

  OCC Monthly Report for September 2015

	CBOE	PHLX	ISE	BATS	ARCA	AMEX
Equity (%)	16.32	17.50	10.53	14.37	11.81	8.89
ETF (%)	15.93	15.02	15.55	10.48	10.14	10.14

- This level of decentralized trading is not an option-only phenomenon. By now, US stocks are regularly traded on 11 exchanges (lit market) and many alternative platforms (dark pools).

Market Participants

- By types,
  - Designated market makers: facilitate trades and provide liquidity.
  - Customers from full service brokerage firms: e.g., hedge funds.
  - Customers from discount brokerage firms: e.g., retail investors.
  - Firm proprietary traders: prop trading desks in investment banks.
- By their activities against the market makers:
  - open buy: buy options to open a new position.
  - open sell: sell/write options to open a new position.
  - close buy: buy options to close an existing position.
  - close sell: sell/write options to close an existing position.

Option Trading Volume by Investor Types

	open buy		open sell		close buy		close sell	
	put	call	put	call	put	call	put	call
Small Stocks								
avg volume	16	53	18	49	8	18	9	26
% Firm Proprietary	7.48	4.46	5.42	4.09	4.42	4.84	3.83	3.75
% Discount Broker	7.35	12.92	9.96	11.97	7.81	11.14	6.74	11.89
% Full-Service Broker	72.61	71.73	75.84	73.66	77.90	72.09	75.96	71.60
Medium Stocks								
avg volume	38	96	36	89	17	39	21	57
% Firm Proprietary	10.87	8.81	9.89	7.62	8.19	8.17	6.76	6.85
% Discount Broker	8.49	12.48	9.38	9.97	8.67	9.34	9.73	12.27
% Full-Service Broker	69.22	67.90	71.38	72.37	71.42	69.89	69.36	68.14
Large Stocks								
avg volume	165	359	135	314	66	159	90	236
% Firm Proprietary	14.45	11.36	13.61	10.14	11.18	9.86	9.19	8.25
% Discount Broker	9.77	13.18	7.83	8.02	7.73	7.55	11.31	13.64
% Full-Service Broker	63.60	64.70	69.68	71.98	68.72	69.95	65.27	65.84
S&P 500 (SPX)								
avg volume	17398	10254	12345	11138	7324	7174	10471	6317
% Firm Proprietary	23.51	34.29	35.71	25.51	32.51	20.05	20.10	28.24
% Discount Broker	4.22	4.19	1.38	1.59	1.48	1.72	4.45	4.78
% Full-Service Broker	58.24	48.16	48.81	59.45	49.75	63.79	59.58	51.72

Source: Pan and Poteshman (2006), CBOE data from 1990 through 2001.

A Nobel-Prize Winning Formula

The Bank of Sweden Prize in Economic Sciences in Memory of Alfred Nobel 1997

"for a new method to determine the value of derivatives"

Robert C. Merton

1/2 of the prize

USA

Harvard University
Cambridge, MA, USA

b. 1944

Myron S. Scholes

1/2 of the prize

USA

Long Term Capital
Management
Greenwich, CT, USA

b. 1941
(in Timmins, ON, Canada)

The Black-Scholes Model

- **The Model:** Let S_t be the time-t stock price, ex dividend. Prof. Black, Merton, and Scholes use a geometric Brownian motion to model S_t:

$$dS_t = (\mu - q)\, S_t\, dt + \sigma\, S_t\, dB_t\, .$$

- **Drift:** $(\mu - q)\, S_t\, dt$ is the deterministic component of the stock price. The stock price, ex dividend, grows at the rate of $\mu - q$ per year:
  - μ: expected stock return (continuously compounded), around 12% per year for the S&P 500 index.
  - q: dividend yield, round 2% per year for the S&P 500 index.
- **Diffusion:** $\sigma\, S_t\, dB_t$ is the random component, with B_t as a Brownian motion. σ is the stock return volatility, around 20% per year for the S&P 500 index.

Brownian Motion

- **Independence of increments:** For all $0 = t_0 < t_1 < \ldots < t_m$, the increments are independent:

$$B(t_1) - B(t_0),\ B(t_2) - B(t_1),\ \ldots,\ B(t_m) - B(t_{m-1})$$

Translating to Finance: stock returns are independently distributed. No predictability and zero auto-correlation $\rho = 0$.

- **Stationary normal increments:** $B_t - B_s$ is normally distributed with zero mean and variance $t - s$.
Translating to Finance: stock returns are normally distributed. Over a fixed horizon of T, return volatility is scaled by $\sqrt{T}$.
- **Continuity of paths:** $B(t)$, $t \geq 0$ are continuous functions of t.
Translating to Finance: stock prices move in a continuous fashion. There are no jumps or discontinuities.

The Model in R_T

- It is more convenient to work in the log-return space:

$$R_T = \ln S_T - \ln S_0\,, \text{ or equivalently, } S_T = S_0 e^{R_T}$$

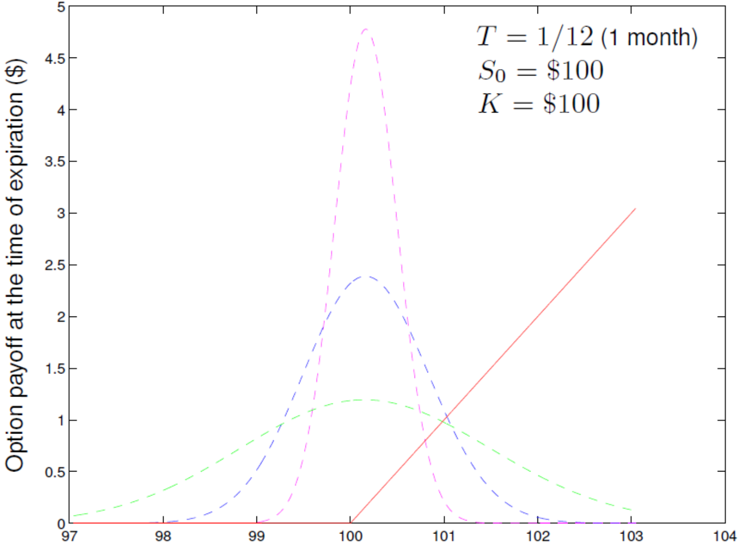
- Using the model for S_T, we get

$$R_T = \left(\mu - q - \frac{1}{2}\sigma^2\right) T + \sigma\sqrt{T}\,\epsilon_T\,,$$

- Most of the terms are familiar to us:
  - $(\mu - q)T$ is the expected growth rate, ex dividend, over time T.
  - $\sigma\sqrt{T}$ is the stock return volatility over time T.
  - ϵ_T is a standard normal (inherited from the Brownian motion).
- The extra term of $-\frac{1}{2}\sigma^2\,T$ is called the Ito's term. It needs to be there because the transformation from S_T to R_T involves taking a log, which is a non-linear (concave) function, of the random variable S_T.

The Payoff of a Call Option: $(S_T - K) \times \mathbf{1}_{S_T > K}$

$\mathbf{1}_{S_T > K} = 1$ if $S_T > K$ and zero otherwise.



Valuation in Finance

- By now, we have a dynamic model for the stock price S_T. We also know the payoff function of a call option: $(S_T - K)\,\mathbf{1}_{S_T > K}$.
- We are now ready to calculate the market value, C_t, of this call option at any given time t before the expiration date T.
- We do so by arbitrage pricing. Recall that in obtaining the CAPM pricing relation, we used *equilibrium pricing*, where mean-variance investors optimize their utility functions, and the equity and bond markets clear.
- In Finance, when it comes to valuation, there are just two approaches: equilibrium pricing and arbitrage pricing.

The Insight of Arbitrage Pricing

- The key insight of arbitrage pricing is very simple: **replication.**
- A security offers me a stream of random payoffs:
  - If I can replicate that cash flow (no matter how random they might be), then the price tag equates the cost of replication.
  - Simple? In reality, it is difficult to find such exact replications.
  - This makes sense: Why do we need a security that can be replicated?
- An option offers a random payoff at the time of expiration T:
  - The most important insight: dynamic replication.
  - The limitation: the replication is done under the Black-Scholes model.
  - The pricing formula is valid if the assumptions of the model are true.

Risk-Neutral Pricing

- Risk-neutral pricing is a widely adopted tool in arbitrage pricing.
- Our model in the return space:

$$\text{P-measure:} \quad R_T = \left(\mu - q - \frac{1}{2}\sigma^2\right) T + \sigma\sqrt{T}\,\epsilon_T\,.$$

- In risk-neutral pricing, we bend the reality by making the stock grow instead at the riskfree rate r:

$$\text{Q-measure:} \quad R_T = \left(r - q - \frac{1}{2}\sigma^2\right) T + \sigma\sqrt{T}\,\epsilon_T^Q$$

- Risk-neutral pricing: cash flows are discounted by the riskfree rate r and expectations are done under the Q-measure:

$$C_0 = E^Q\left(e^{-rT}(S_T - K)\mathbf{1}_{S_T > K}\right)$$

Pricing a Stock

- Consider the S&P 500 index and assume zero dividend $q = 0$. The index's final payoff is S_T. How much are you willing to pay for it today? Of course, S_0.
- Under P-measure:

$$e^{-\mu T} E^{\mathrm{P}}(S_T) = e^{-\mu T}\, S_0 e^{\mu T} = S_0$$

- Under Q-measure:

$$e^{-r T} E^{\mathrm{Q}}(S_T) = e^{-rT}\, S_0 e^{rT} = S_0$$

- Pricing using a Risk-neutral investor:

$$e^{-r T} E^{\mathrm{P}}(S_T) = e^{-rT}\, S_0 e^{\mu T} = S_0\, e^{(\mu - r) T}$$

- Risk-neutral pricing does not mean pricing using a risk-neutral investor.

Pricing a Call Option

- Let C_0 be the present value of a European-style call option on S_T with strike price K. Using risk-neutral pricing:

$$C_0 = E^Q\left(e^{-rT}(S_T - K)\mathbf{1}_{S_T>K}\right)$$
$$= \boxed{e^{-rT}\, E^Q\left(S_T \mathbf{1}_{S_T>K}\right)} - \boxed{e^{-rT}\, K\, E^Q\left(\mathbf{1}_{S_T>K}\right)}$$

- Let's go directly to the solution (again assume $q = 0$ for simplicity):

$$C_0 = \boxed{S_0\, N(d_1)} - \boxed{e^{-rT} K N(d_2)},$$

  where $N(d)$ is the cumulative distribution function of a standard normal.

- Comparing the terms in blue, we have $N(d_2) = E^Q(\mathbf{1}_{S_T>K})$, which is $\text{Prob}^Q(S_T > K)$, the probability that the option expires in the money under the Q-measure.
- Comparing the terms in green: $N(d_1) = e^{-rT} E^Q\left(\frac{S_T}{S_0}\mathbf{1}_{S_T>K}\right)$.

Understanding d_2 and d_1:

$$d_1 = \frac{\ln\left(S_0/K\right) + \left(r + \sigma^2/2\right)\, T}{\sigma\sqrt{T}}\,; \quad d_2 = \frac{\ln\left(S_0/K\right) + \left(r - \sigma^2/2\right)\, T}{\sigma\sqrt{T}}$$

- The model for S_T under Q-measure is $S_T = S_0\, e^{R_T}$ with

$$\text{Q-measure:} \quad R_T = \left(r - \frac{1}{2}\sigma^2\right) T + \sigma\sqrt{T}\,\epsilon_T^{\mathrm{Q}}$$

- We can verify that $N(d_2)$ indeed gives us $\text{Prob}^Q\left(S_T > K\right)$: the probability that the option expires in the money under the Q-measure.
- What about $N(d_1)$? With $E\left(S_T \mathbf{1}_{S_T > K}\right)$, it calculates the expectation of S_T only when $S_T > K$. This calculation is not required for exams.
- If you like, you can think of $N(d_1)$ as $\text{Prob}^{QQ}\left(S_T > K\right)$,

$$\text{QQ-measure:} \quad R_T = \left(r + \frac{1}{2}\sigma^2\right) T + \sigma\sqrt{T}\,\epsilon_T^{\mathrm{QQ}}$$

The Black-Scholes Formula:

- The Black-Scholes formula for a call option (bring dividend back),

$$C_0 = e^{-qT} S_0 \, N(d_1) - e^{-rT} K \, N(d_2)$$

$$d_1 = \frac{\ln\left(S_0/K\right) + \left(r - q + \sigma^2/2\right) T}{\sigma\sqrt{T}} \, , d_2 = \frac{\ln\left(S_0/K\right) + \left(r - q - \sigma^2/2\right) T}{\sigma\sqrt{T}}$$

- Put/call parity is model free. Holds even if the Black-Scholes model fails,

$$C_0 - P_0 = e^{-q\,T} S_0 - e^{-r\,T} K \, .$$

  Empirically, this relation holds well in the data and is similar in spirit to the arbitrage activity between the futures and cash markets.
- Using put/call parity, the Black-Scholes pricing formula for a put option is:

$$\begin{aligned} P_0 &= -e^{-qT} S_0 \, \left(1 - N(d_1)\right) + e^{-r\,T} K \, \left(1 - N(d_2)\right) \\ &= -e^{-qT} S_0 \, N(-d_1) + e^{-r\,T} K \, N(-d_2) \end{aligned}$$

At-the-Money Options (assume $q = 0$)

- For an at-the-money option, whose strike price is $K = S_0\, e^{r\,T}$

$$C_0 = P_0 = S_0 \left[N\left(\frac{1}{2}\sigma\sqrt{T}\right) - N\left(-\frac{1}{2}\sigma\sqrt{T}\right)\right]$$

- Recall that $N(d)$ is the cdf of a standard normal,

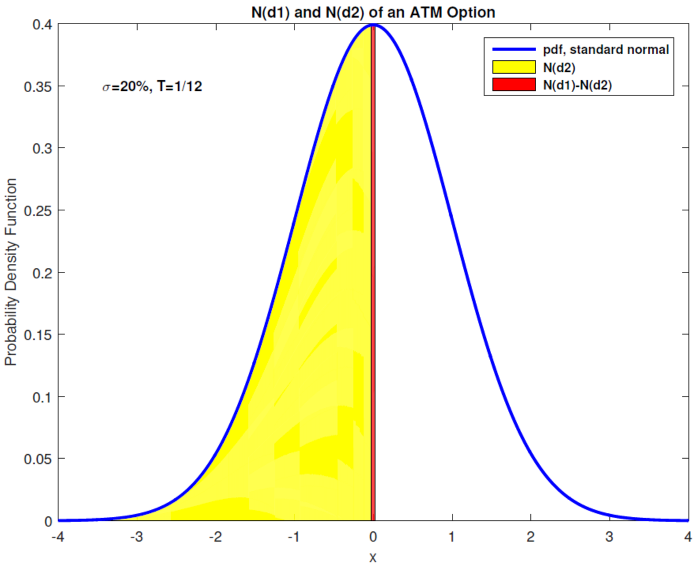
$$N(d) = \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx$$

- So the pricing formula can be further simplified to

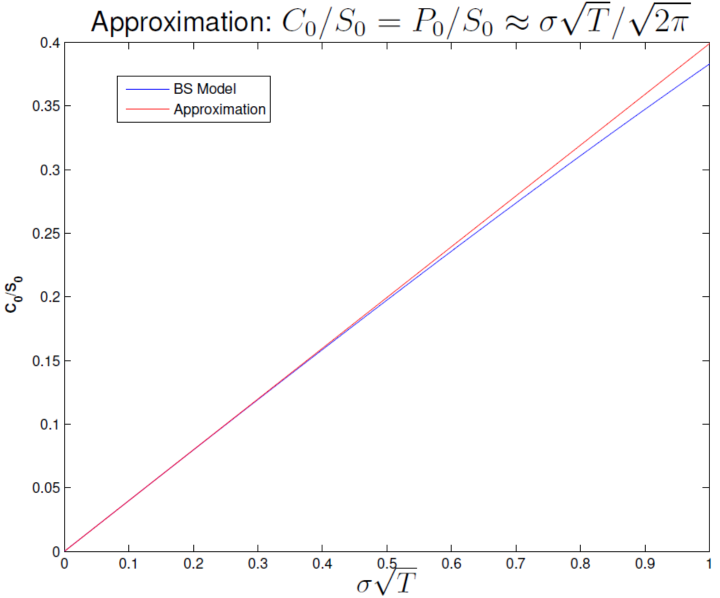
$$\frac{C_0}{S_0} = \frac{P_0}{S_0} = \int_{-\frac{1}{2}\sigma\sqrt{T}}^{\frac{1}{2}\sigma\sqrt{T}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx \approx \frac{1}{\sqrt{2\pi}}\, \sigma\sqrt{T}\,,$$

  which works well for small $\sigma\sqrt{T}$. For large $\sigma\sqrt{T}$ (volatile markets or long-dated options), non-linearity becomes important and this approximation is imprecise.

ATM Options: $d_1 = \frac{1}{2}\sigma\sqrt{T}$ and $d_2 = -\frac{1}{2}\sigma\sqrt{T}$



ATM Options as a Linear Contract on $\sigma\sqrt{T}$



The Black-Scholes Option Implied Volatility:

- At time 0, a call option struck at K and expiring on date T is traded at C_0. At the same time, the underlying stock price is traded at S_0, and the riskfree rate is r.
- If we know the market volatility σ at time 0, we can apply the Black-Scholes formula:

$$\boxed{C_0^{\text{Model}} = \text{BS}(S_0, K, T, \sigma, r, q)}$$

- Volatility is something that we don't observe directly. But using the market-observed price C_0^{Market}, we can back it out:

$$C_0^{\text{Market}} = C_0^{\text{Model}} = \text{BS}(S_0, K, T, \sigma^I, r, q)\,.$$

- If the Black-Scholes model is the correct model, then the Option Implied Volatility σ^I should be exactly the same as the true volatility σ.

Example 1: Option Quotes from CBOE (Oct 2011):

SPX (S&P 500 INDEX) **1200.86** **-23.72**

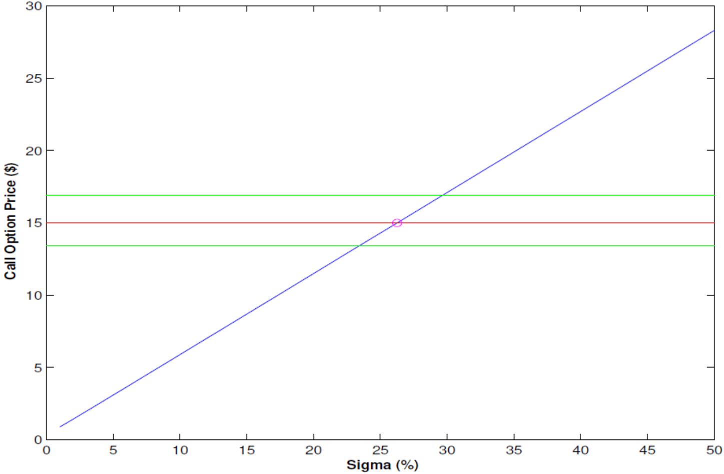
Oct 17, 2011 @ 18:32 ET

Calls	Last Sale	Net	Bid	Ask	Vol	Open Int		Puts	Last Sale	Net	Bid	Ask	Vol	Open Int
SPX1122J1195-E	20.40	-15.55	15.90	19.50	479	3011	11 Oct 1195.00	SPX1122V1195-E	13.50	+7.00	11.70	14.50	268	2062
SPX1122J1200-E	15.00	-16.70	13.40	16.90	22631	130393	11 Oct 1200.00	SPX1122V1200-E	14.40	+7.40	14.80	17.20	30446	119114
SPX1122J1205-E	12.00	-16.40	11.20	13.20	491	5139	11 Oct 1205.00	SPX1122V1205-E	15.50	+7.60	16.60	20.10	139	887
SPX1122J1210-E	12.00	-12.50	9.00	10.00	2503	15668	11 Oct 1210.00	SPX1122V1210-E	18.20	+9.00	19.30	22.20	3980	8298
SPX1119K1195-E	41.20	-1.10	39.70	43.10	13	248	11 Nov 1195.00	SPX1119W1195-E	40.20	+10.20	38.00	41.20	4	812
SPX1119K1200-E	38.00	-11.50	36.50	40.20	9753	70309	11 Nov 1200.00	SPX1119W1200-E	42.00	+13.00	40.00	43.50	45100	51369
SPX1119K1205-E	37.10	-10.50	33.60	37.30	207	315	11 Nov 1205.00	SPX1119W1205-E	41.50	+7.70	42.50	45.90	63	235
SPX1119K1210-E	35.10	-9.30	30.80	34.60	292	1908	11 Nov 1210.00	SPX1119W1210-E	45.00	+12.50	44.80	48.30	1529	1872

- The current index level is at 1200.86, down -23.72 (or 1.94%) from the day before.
- The *near-term* and *near-to-the-money* call option is the "SPX1122J1200-E" contract, which is traded at \$13.40 bid and \$16.90 ask. The mid-quote price is \$15.15. (The bid/ask spread is \$3.5 and the percentage bid/ask spread is 23%.)
- The last transaction price is \$15.00, down \$16.70 (or 52.68%) from the day before. There were 22,631 such contracts traded and the open interest (or number of option contracts outstanding) is 130,393.

Back out the option implied volatility:

S_0	K	T	r	q	σ
1200.86	1200	5/365	0.0024	0.02	?



Other Contracts

	C_0 or P_0	S_0	K	T	r	q	σ (%)
Call	15.00	1200.86	1200	$5/365$	0.0024	0.02	26.25
Put	14.40	1200.86	1200	$5/365$	0.0024	0.02	26.20
Call	38.00	1200.86	1200	$34/365$	0.0024	0.02	26.41
Put	42.00	1200.86	1200	$34/365$	0.0024	0.02	28.39

Data Synchronization Issues

- Notice that in our calculation of the Black-Scholes implied volatility, we assume that the index options are traded at exactly the same time as the underlying index (which was marked at 1200.86).
- In practice, they are marked as "Last Sale," and we actually do not know when the last sale took place for each option.
- So to be precise, we need to have the time stamp of each option transaction and then use the time stamp to retrieve the underlying index level at exactly the same time. CBOE offer such transaction level data, but it is expensive.
- The trading hours for SPX options are 8:30am-3:15pm Chicago time, while the NYSE hours are 9:30am-4:00pm New York time.

Put/Call Parity

- Notice also that put/call parity is violated by close to 4 dollars for the November 1200 contracts. In this market, put/call parity holds pretty well. In other words, it has to be a data issue.
- If we are confident that the paired call/put options (the same time to expiration and the same strike price) are traded at exactly the same time, then we can actually use put/call parity to back out the underlying index level (instead of relying on the underlying market).
- According to CBOE, the VIX index measured at close is 33.39 and the VXO index measured at close is 31.40.

Example 2: Option Quotes from CBOE (Oct 2005):

.SPX **1195.90** **+4.41**

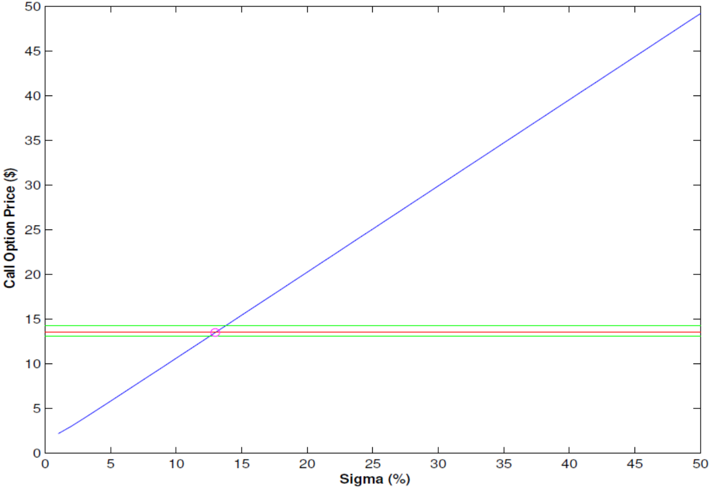
Oct 07, 2005 @ 17:31 ET (Data 15 Minutes Delayed)

Calls	Last Sale	Net	Bid	Ask	Vol	Open Int	Puts	Last Sale	Net	Bid	Ask	Vol	Open Int
05 Oct 1190. (SPT JR-E)	17.30	+1.80	16.50	17.40	638	1050	05 Oct 1190. (SPT VR-E)	9.50	-3.00	9.40	10.10	3474	35434
05 Oct 1195. (SPT JS-E)	13.50	+1.00	13.10	14.30	3524	2212	05 Oct 1195. (SPT VS-E)	11.50	-3.00	11.00	12.20	2091	8914
05 Oct 1200. (SZP JT-E)	11.00	+1.00	10.40	11.30	1875	24502	05 Oct 1200. (SZP VT-E)	14.00	-2.60	13.50	14.00	8467	53595
05 Oct 1205. (SZP JA-E)	8.60	+0.70	8.00	8.90	1252	2875	05 Oct 1205. (SZP VA-E)	18.00	-3.00	15.80	17.00	428	15908

- The current index level is at 1195.90, up 4.41 from the day before..
- The *near-term* and *near-to-the-money* call option is the "05 Oct 1195" contract, which is traded at \$13.10 bid and \$14.30 ask. The mid-quote price is \$13.70.
- The last transaction price is \$13.50, up \$1 from the day before.
- There were 3524 such contracts traded and the open interest (or number of option contracts outstanding) is 2212.
- The total dollar trading volume for this Oct contract is $13.7 \times 100 \times 3524 = \$4.8\text{m}$.

Back out the option implied volatility:

S_0	K	T	r	q	σ
1195.90	1195	15/365	0.04	0.02	?



Some Details:

- October contracts expire on Oct. 22, which is the Saturday after the third Friday of the month. The option trading took price on Oct. 7, which is 15 calendar days away from the expiration date.
- The assumptions of $r = 4\%$ and $q = 2\%$ are approximations. For longer-dated options, assumptions on the riskfree rate and dividend yield become more important.
- In Excel, you can back out the implied volatility using a solver.
- We can perform the same exercise on put options to obtain put option implied volatility.

Bring the Black-Scholes Model to the Data

- The key assumptions of the model: constant volatility, continuous price movements, and log-normal distribution.
- The data: S&P 500 index options of different levels of moneyness and time to expiration.
- The basic tool: the Black-Scholes Option Implied Volatility.