Class 3: Calculating Standard Errors

Empirical Asset Pricing, Fall 2020

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Estimating the Mean of a Time-Series

- Moment condition $E\left[h\left(R_t, \mu_0\right)\right] = 0$ with moment h:

$$h\left(R_t, \mu\right) = R_t - \mu\,, \text{ for any } \mu \in R\,.$$

- Population mean of the moment, for any $\mu \in R$:

$$H_0(\mu) = E\left[h\left(R_t, \mu\right)\right] = E\left(R_t - \mu\right)$$

- Sample counterpart:

$$H_T(\mu) = \frac{1}{T}\sum_{t=0}^{T} h\left(R_t, \mu\right) = \frac{1}{T}\sum_{t=0}^{T}\left(R_t - \mu\right)$$

Calculating Standard Errors from First Principles

- Mean-value theorem, for some b^* between b_0 and b_T,

$$H_T(b_T) - H_T(b_0) = \left.\frac{\partial H_T}{\partial b}\right|_{b=b^*} (b_T - b_0)$$

$$(r \times 1) \qquad (r \times q) \qquad (q \times 1)$$

  where there are r moments and q parameters.

- Multiply both sides by

$$\left(\left.\frac{\partial H_T(b)}{\partial b}\right|_{b_T} \right)^T W_T$$

- By definition, b_T minimizes $H_T(b)^T W_T H_T(b)$, so

$$\left(\left.\frac{\partial H_T(b)}{\partial b}\right|_{b_T} \right)^T W_T \, H_T(b_T) = 0$$

Calculating Standard Errors from First Principles

- We are left with

$$-\left(\left.\frac{\partial H_T(b)}{\partial b}\right|_{b_T}\right)^T W_T\, H_T(b_0) = \left(\left.\frac{\partial H_T(b)}{\partial b}\right|_{b_T}\right)^T W_T\, \left.\frac{\partial H_T}{\partial b}\right|_{b=b^*} (b_T - b_0)$$

- Assuming the $q \times q$ matrix in front of $b_T - b_0$ is invertible, we have

$$b_T - b_0 = -\left[\left(\left.\frac{\partial H_T(b)}{\partial b}\right|_{b_T}\right)^T W_T\, \left.\frac{\partial H_T}{\partial b}\right|_{b=b^*}\right]^{-1} \left(\left.\frac{\partial H_T(b)}{\partial b}\right|_{b_T}\right)^T W_T\, H_T(b_0)$$

- Asymptotically, when T goes to ∞, we have b_T converges in probability to b_0 and $\sqrt{T}\,(b_T - b_0)$ converges in distribution to normal asymptotic variance-covariance matrix

$$\Omega = (d_0^T\, W_0\, d_0)^{-1} d_0^T\, W_0\, \Sigma_0\, W_0\, d_0\, (d_0^T\, W_0\, d_0)^{-1}$$

Calculating Standard Errors from First Principles

- Σ_0 is the asymptotic variance-covariance matrix, dimension $r \times r$, of the sample mean of the moments $H_T(b_0)$:

$$\Sigma_0 = \lim_{T\to\infty} TE\left(H_T(b_0)H_T(b_0)^T\right)$$

- d_0 is the derivative matrix of the sample mean:

$$d_0 = E\left(\left.\frac{\partial H_T(b)}{\partial b}\right|_{b_0}\right)$$

- W_0 is the weighting matrix
  - In GMM, $W_0 = \Sigma_0^{-1}$. In this case, $\Omega = \left[D^T\Sigma_0^{-1}D\right]^{-1}$
  - Setting $W_0 = I$, we pick $r = q$ moments to estimate q parameters and D has to be invertible. In this case $\Omega = D^{-1}\Sigma_0(D^T)^{-1} = \left[D^T\Sigma_0^{-1}D\right]^{-1}$

Estimating Other Moments

- Standard deviation $\sigma = \sqrt{m_2}$:

$$h\left(R_t, m_1, m_2\right) = \begin{pmatrix} R_t - m_1 \\ \left(R_t - m_1\right)^2 - m_2 \end{pmatrix}$$

- Skewness:

$$\frac{E(R_t - m_1)^3}{m_2^{3/2}}$$

- Kurtosis:

$$\frac{E(R_t - m_1)^4}{m_2^2}$$

- Correlation:

$$\frac{E\left[(X_t - m_1^X)(Y_t - m_1^Y)\right]}{\sqrt{m_2^X m_2^Y}}$$

Regression: $Y = X^T\beta + u$

- Sample mean of the moments:

$$H_T(\beta) = \frac{1}{T}\sum_t X\left(Y - X^T\beta\right)$$

- Derivative matrix:

$$\frac{\partial H_T}{\partial \beta} = -\frac{1}{T}\sum_t X X^T\,; d_0 = -E(X\,X^T)$$

- Asymptotic variance-covariance matrix of H_T:

$$\Sigma_0 = \lim_{T\to\infty} T E\left(H_T\,H_T^T\right) = \lim_{T\to\infty}\frac{1}{T}E\left(\sum X(Y - X^T\beta)\left(\sum X(Y - X^T\beta)\right)^T\right)$$

- Standard errors: OLS, White, Newey-West, clustered by time and firm (for panel regressions).