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Empirical Asset Pricing, Fall 2019 | November 27, 2019

Classes 14-16: Treasury Bonds

The following was written for the master-level class I taught at MIT Sloan a few years ago. If time permits, I'll revise the content to make it more suitable for a PhD level class. The first part of the note was originally written in November 2015. In many places, "right now" means Fall 2015. Just a quick update on the numbers: as of November 26, 2019, the three-month Treasury yield is at 1.54%, the 10-year yield is at 1.75%, and the 30-year at 2.19%.

1 From Equity to Fixed Income

• **Vehicles for Risk:** Moving from equity to options to bonds and, later, to OTC derivatives, there is always one thing in common: each market is a vehicle for risk. The nature and origin of the risk might vary from one market to the other, but our approach to risk remains the same.

We plot the time-series data to see how it varies over time. We map the historical experiences into a distribution and use it as a basis to envision future scenarios. Thinking of the future in a static fashion as one fixed future date, we employ random variables to model the distribution at this future date (e.g., the CAPM). Thinking of the future in a dynamic fashion as a path leading into the future, we use stochastic processes to model the random paths (e.g., Black-Scholes). Either way, we use these models to price the risk involved, taking into account not only the likelihood and magnitude of the risk, but also investors' attitudes to the risk. After this is done, we go back to the data to see how well our model performs. Very often, the data surprises us. In this process of model meeting the data, new insights arise.

• **Relating one to the other:** You might also notice that, in Finance, we keep ourselves busy by relating one thing to the other. For example, in the equity market, we relate the individual stock returns R_t^i to the contemporaneous returns of the market portfolio R_t^M. The pricing of an individual stock is done through the pricing of the market:

$$E(R_t^i) - r_f = \beta^i \left(E(R_t^M) - r_f \right) .$$

By doing so, we narrow our attention down to one risk factor: the market portfolio. In the crowd of thousands of stocks, your eyes are on this one and one thing only, and everything else fades into the background.

In options, we relate the time-t option price C_t to two things: the price of the underlying stock S_t and the volatility of the underlying stock σ. The relation between C_t and S_t is useful, but what really makes options unique is the relation between C_t and σ. This is especially important when we step outside of the Black-Scholes model and allow σ_t to vary over time: now options are unique vehicles for the risk in σ_t. This is why I asked you to pay special attention to this approximation for an ATM option:

$$C_t/S_t = P_t/S_t \approx \frac{1}{\sqrt{2\pi}} \sigma \sqrt{T} .$$

Now we are studying the fixed-income market, which is large and important, encompassing products such as Treasury bonds (\$12.5tn), mortgage-backed securities (\$8.7tn), corporate bonds (\$7.8tn), Muni (\$3.6tn), money market funds (\$2.9tn), agency bonds (\$2.0tn), and asset-backed securities (\$1.3tn). The numbers in parentheses are amount outstanding as of end 2014. At the center of our attention is the risk that is common to all of these products: interest rate fluctuations. Not one interest rate, but many: one for each maturity. Putting them together, we have a yield curve. In Finance, there is no other risk that is more important than this yield curve risk. It is fundamental to everything we do in Finance. It is the basis from which all other discount rates are calculated.

In dealing with this risk, we prefer to work in the yield space because it is more convenient, but the profit/loss happens in the dollar space. As a result, we will be busy relating one thing to the other again. This gives rise to concepts such as duration and convexity. An outsider might look at these funny names and accuse people in Finance of creating unnecessary concepts so as to confuse and take advantage of those who know less about finance. There might be such practices going on elsewhere on Wall Street, but concepts such as duration and convexity and Black-Scholes implied vol are created out of necessity. I cannot imagine myself navigating the bond market without having tools like duration and convexity.

• **Focus on What's Important:** In talking about beta in equity, implied-vol in options, and duration and convexity in bonds, my intention is to remind you to focus on what's important.

Often, I notice that some students have the tendency to focus on the small and trifling things first before trying to digest the more important message. When you look at a tree, your attention goes first to the overall structure and shape, not to a small offshoot from a branch of the tree (unless there is a cat sitting there). If you are drowning, you grab the nearest and largest lifesaver available; you don't stop to examine the color or the make of the lifesaver. Nor do you question whether or not the lifesaver is made of sustainable materials.

So please, go for the important concept first. Only after you understand these concepts really well, then you have the luxury in digging into the minute details. Of course, ideally, you would like to be good at both: big-picture and rigor. But in the process of learning, it makes sense to go after the big picture first.

While I am on this topic, let me also add that you should always bring your common sense back to anything you do in Finance. For example, it is very easy to get lost when working on a project. Sooner or later, the model and the spreadsheet become the boss and you the slave. Use your common sense. Don't invest in any fancy models or techniques until you have a very clear view of why you need them. Otherwise, it will be garbage in and garbage out. In the process, you might manage to impress yourself and a few others with the fancy techniques and models. But in truth, it is mostly confusion.

The same thing applies to a professor. If, after each class, I make you more confused than before, then I am not doing a good job in teaching the materials. That is why I am writing the lecture notes, to give myself ... a second chance.

• **In the Return Space:** Coming back to our main topic, I list in Table 1 summary statistics of equity (the CRSP value-weighted index) and bond returns using monthly data from 1942 through 2014. In the second panel of the table, I also report the numbers for the more recent period from 1990 through 2014.

For the sample period from 1942 through 2014, the average monthly return of the US stock market is 1.03% and the volatility is about 4.16%. In annualized terms, the average return is 12.33% and the volatility is 14.4%. (The 20% annual volatility number we've been using includes the great depression.) For the same period, the average return of a 10-year bond is about 47 basis points per month and the volatility is about 2%. Not surprisingly, with decreasing maturity (and duration), both the average return and volatility decrease for shorter maturity bonds. The one-month TBill has an average return of 32 basis points per month, and an average yield of $0.32\% \times 12 = 3.84\%$. The monthly volatility of the one-month Treasury bill is 0.26%, which is only a small fraction of that in the stock market (4.16%).

Table 1 also reports the best and worst one-month returns for each of the securities. Not surprisingly, the stock market is the most risky with the largest range of minimum and

Table 1: Monthly Equity Returns and Bond Returns

Monthly 1942-2014	mean (%)	std (%)	Sharpe ratio	min (%)	max (%)	correlation with Stock	TBill	10Y
Stock	1.03	4.16	0.17	-21.58	16.81	1.00	-0.05	0.10
10Y Bond	0.47	2.00	0.08	-6.68	10.00	0.10	0.12	1.00
5Y Bond	0.46	1.38	0.10	-5.80	10.61	0.07	0.19	0.90
2Y Bond	0.42	0.77	0.13	-3.69	8.42	0.08	0.37	0.76
1Y Bond	0.40	0.50	0.16	-1.72	5.61	0.08	0.59	0.62
1M TBill	0.32	0.26		-0.00	1.52	-0.05	1.00	0.12
CPI	0.31	0.45		-1.92	5.88	-0.07	0.26	-0.07
Monthly 1990-2014	mean (%)	std (%)	Sharpe ratio	min (%)	max (%)	correlation with Stock	TBill	10Y
Stock	0.87	4.22	0.15	-16.70	11.41	1.00	0.01	-0.06
10Y Bond	0.57	1.99	0.16	-6.68	8.54	-0.06	0.07	1.00
5Y Bond	0.50	1.24	0.20	-3.38	4.52	-0.10	0.15	0.93
2Y Bond	0.39	0.54	0.26	-1.30	2.07	-0.11	0.41	0.74
1Y Bond	0.33	0.31	0.26	-0.33	1.31	-0.03	0.72	0.51
1M TBill	0.25	0.19		-0.00	0.68	0.01	1.00	0.07
CPI	0.21	0.34		-1.92	1.22	-0.04	0.18	-0.16

maximum. During the sample period from 1942 to 2014, the worst one-month return was -21.58%, which happened in October 1987.

Also reported are the correlations between the stock returns and the bond returns. The correlation between these two markets is very weak and is also unstable. The correlation between stock and 10-year bond is 10% for the sample from 1942 through 2014 and -6% for the more recent sample from 1990 through 2014. Unlike the low correlation between stock and bond, the correlations between the bond returns are relatively high. The closer the maturity (e.g., 10Y and 5Y), the higher the correlation. We will come back and the investigate this issue in our next class when we do PCA (Principal Component Analysis) on bonds.

It is also interesting to see that the correlations between inflation (CPI) and the stock returns and 10Y are low and slightly negative. The correlation between inflation and the 1M Tbill is about 26% for the entire sample and 18% for the more recent sample. Note that we are working with nominal interest rate, which is the sum of real interest rate and inflation. As you can see from Table 1, the average inflation is close to the 1-month Treasury bill, but slightly lower, implying that the real interest rate is on average positive.

• **The Cycle of Hot and Cold:** Using the average return of the one-month Treasury bill as the riskfree rate, we can calculate the Sharpe ratios of the equity and bond returns.

From this perspective, bonds have been more attractive (higher average return and lower volatility) for the more recent sample period from 1990 through 2014.

In fact, from the mid 1980s to today, the bond market condition has been quite favorable. The interest rates have been decreasing from the double digits in the early 1980s to today's near-zero. Some call it a 30-year bull market run. In addition to the favorable market condition, we have also seen the rise of MBS, junk bonds, OTC derivatives, asset-backed securities, all of which add to the business of fixed-income desks in investment banks.

When Michael Lewis joined the training program in Salomon in 1985, the bond market was just getting hot, driven by the profitability in bonds. In 1986, other firms like Goldman Sachs were catching up with Salomon's bond expertise by hiring people away from Salomon (See, for example, *Money and Power* by Cohan). Within Salomon, as described in Michael Lewis' book, *Liar's Poker*, an entertaining (maybe too entertaining) book, the desired location was to be on a bond desk. Equity was looked down up, and "Equity in Dallas" was the equivalent of Siberia.

But only ten years prior to that, bond was not at hot and equity was the place to go. Quoting Michael Lewis,

> That, anyway, is what I was told. It was hard to prove any of it because the only evidence was oral. But consider the kickoff chuckle to a speech given to the Wharton School in March 1977 by Sidney Homer of Salomon Brothers, the leading bond analyst on Wall Street from the mid-1940s right through to the late 1970s. "I felt frustrated," said Homer about his job. "At cocktail parties lovely ladies would corner me and ask my opinion of the market, but alas, when they learned I was a bond man, they would quietly drift away."
>
> Or consider the very lack of evidence itself. There are 287 books about bonds in the New York Public Library, and most of them are about chemistry. The ones that aren't contain lots of ugly numbers and bear titles such as *All Quiet on the Bond Front*, and *Low-Risk Strategies for the Investor*. In other words, they aren't the sort of page turners that moisten your palms and glue you to your seat. People who believe themselves of social consequence tend to leave more of a paper trail, in the form of memoirs and anecdotiana. But while there are dozens of anecdotes and several memoirs from the stock markets, the bond markets are officially silent. Bond people pose the same problem to a cultural anthropologist as a nonliterate tribe deep in the Amazon.

By now, bond people are certainly not the equivalent of a nonliterate tribe deep in the Amazon. In fact, if you search Amazon for books on Finance, many of them were written by bond traders. So is this endless cycle of being hot and cold, in and out of favor. Whatever that can go up certainly has the potential to come down. The moment something is in favor marks the beginning of its decline.

Right now (Fall 2015), the interest rate is at a level as low as it can ever be, and the 30-year bull run in the bond market is approaching to an end. Most likely, the Fed will raise the Fed fund rate in its December FOMC meeting this year (Fall 2015). Inferring from the pricing in Fed fund futures, there is a 70% likelihood of a Fed hike at its December 15-16 meeting (Fall 2015). So we will know the result before our final exam on December 17 (Fall 2015).

In the mutual fund world, the famous bond fund, Pimco's Total Return, is a good representation of this cycle of bull and impending bear. As shown in Figure 1, the first observation of Pimco (Total Return Fund, Institutional Class) in my data was at the end of June 1987 with a total net asset value of $12.8 million. From 1987 to 2013, the fund, benefited from the favorable bond market condition, was in a steady ascend, reaching to its peak ($182.8 billion) in April 2013. This grow in the size of a mutual fund has two component: the market performance and fund flows. So the growth from $18 million to $182.8 billion was a combination of both. As we know, in the mutual fund word, flow chases performance. So the favorable condition in the bond market has a lot to do with the growth.

In recent months, the size of the fund has been decreasing quite rapidly. Figure 2 plots the total net asset value for all four classes of the fund. Of course, if you have been following the news since 2014, you would know that the internal powerful struggle and the clash of personalities also contributed to the fund outflow. But the clash of personality probably would not have escalated to such a degree had the bond market condition been favorable.

Also note that the plot is in log-scale, in an effort to damp the high growth rate. If it were plotted in a linear plot, the ups and downs would have been even more dramatic.

As another example of the force of the overall market condition versus the skills of an individual fund manager, I plot in Figure 1 the total net asset value of the once famous equity fund, Fidelity Magellan. The fund shows up in my data since May 1963 but the first reported total net asset value in my data was $6.5 million in December 1967. By December 1975, the fund was smaller at $5.4 million, most likely due to the bear market of 1973-74. In June 1976 Peter Lynch took over the fund. From 1976 to 1990, under Peter Lynch's management, the fund grew in size as well as in fame. After Peter Lynch's retirement in May 1990, the fund kept growing, thankful to the bull market of the late 1990s. The fund grew to its peak ($109.8 billion) in August 2000, and then started its decline after the Internet bubble burst. Right now (Fall 2015), it is a $14 billion fund, roughly the size when Peter Lynch retired from the fund in May 1990.

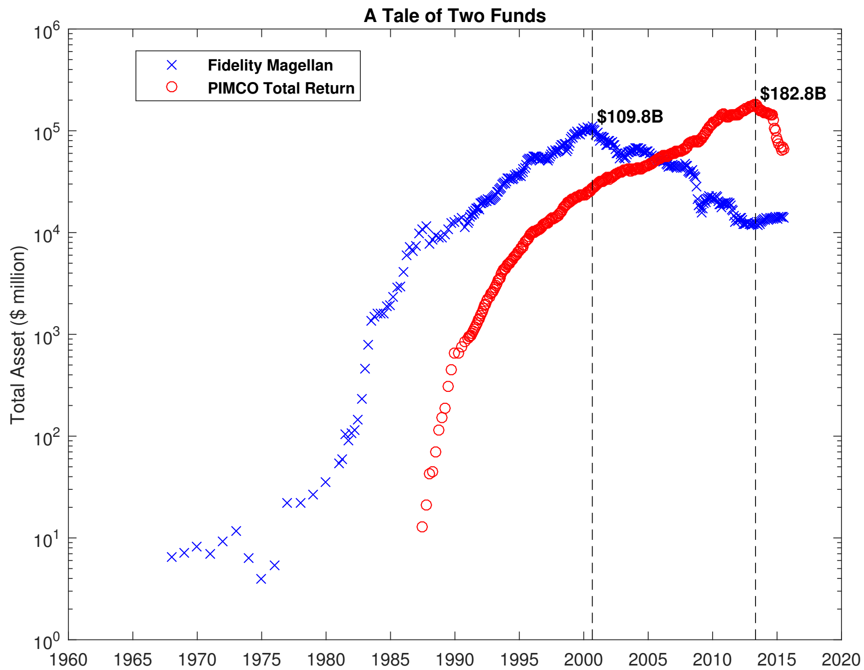

Figure 1: Total Net Asset Value, Fidelity Magellan and Pimco Total Return.

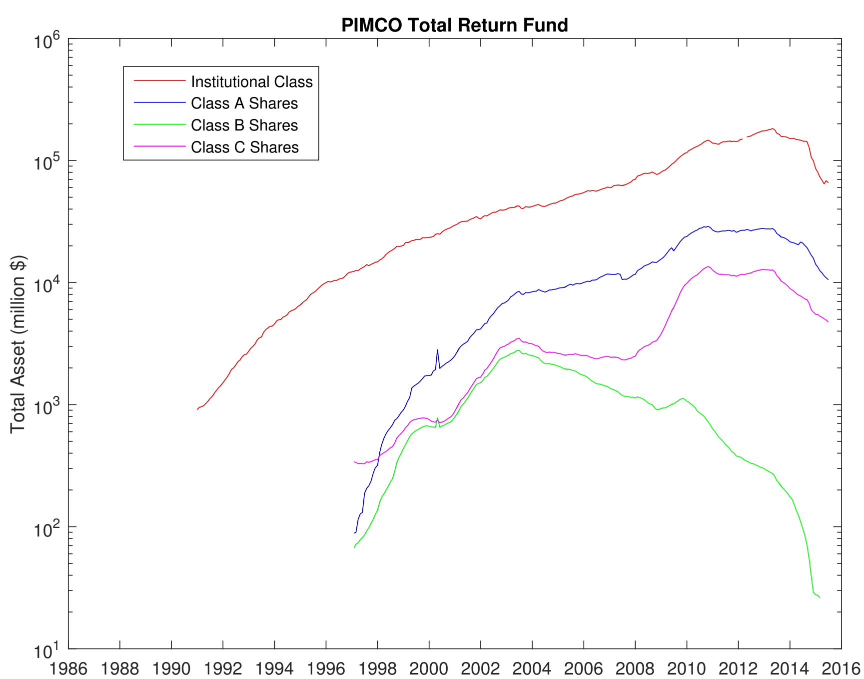

Figure 2: Total Net Asset Value, Pimco Total Return Fund.

Cycles like those in Figure 1 are part and parcel of the financial markets. Such forces in financial markets should be humbling for any human being, no matter how successful this person might be. To attribute one's success entirely to one's talent is pure arrogance and ignorance. If you have not read the recent stories surrounding Bill Gross (the co-founder of Pimco), I would suggest that you do. At some point in your life, you might get lucky and become successful. Try not to let your ego drive you too far. There are no worse enemies in your life than your own ego. In fact, your ego is you only enemy.

2 Bond Price and Yield: Duration and Convexity

• **Bond Price P and Yield to Maturity y:** A Treasury yield curve involves Treasury bonds, notes, and bills. Treasury notes are issued in terms of 2, 3, 5, 7, and 10 years; Treasury bonds are issued at 30 years. A Treasury bond issued 25 years ago would have 5 years to maturity, same as a newly issued 5-year notes. But the coupon rates of the two bonds are different. Coupon bearing bonds are issued at par, making the coupon rate close to the yield to maturity at the time of issuance. Given the current low interest rate environment, the 30-year bond issued 25 years ago has a coupon rate that is higher than the newly issued 5 year notes. It is therefore a premium bond. There are also differences in liquidity, which we will talk about later.

Throughout the fixed-income classes, I'll not make a distinction between notes and bonds and will refer to them simply as bonds. I'll use the notation of P_t as the bond price at time t, and $y_t\%$ as the yield to maturity at time t. At issuance, a Treasury bond is defined by the following parameters: **face value** = \$100; **coupon rate** = c; **maturity** = T years. These parameters are fixed throughout the life of the bond and will not change. Treasury bonds pay coupon semi-annually, and, at issuance, the coupon rate c is chosen so that the bond is priced at par with $P = \$100$. As a result, the yield to maturity y (semi-annual compounding) equals to the coupon rate c when the bond was first issued.

Later, with the fluctuations in interest rates, both P and y will change. There is a deterministic relation between the two:

$$P = \sum_{n=1}^{2T} \frac{\frac{c}{2} \times 100}{\left(1 + \frac{y}{2}\right)^n} + \frac{100}{\left(1 + \frac{y}{2}\right)^{2T}} \,, \tag{1}$$

where both c and y are expressed in percentage. So an increasing interest rate environment after the issuance of the bond is bad news for long-only bond investors: P decreases with increasing y and the bond will be in discount ($P < \$100$). Conversely, a decreasing interest rate environment is good news such a long-only bond investor: P increases with decreasing

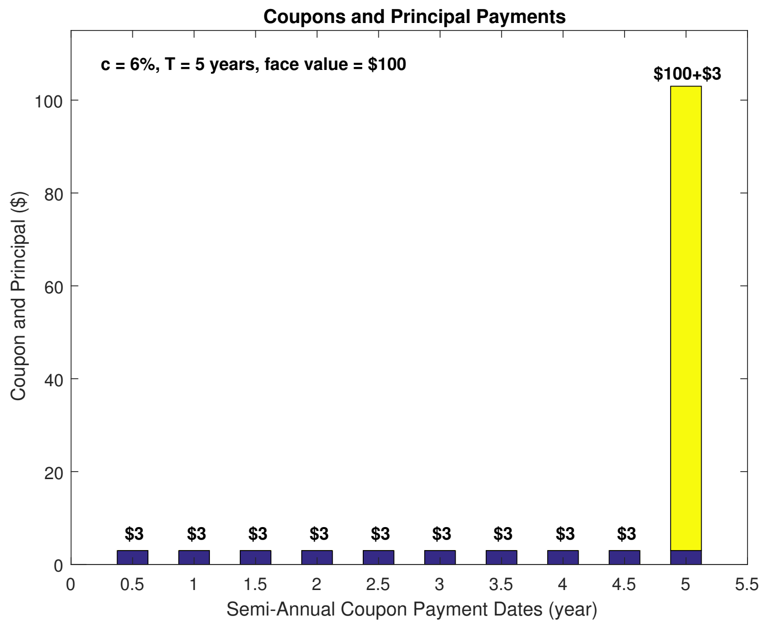

Figure 3: Coupon and Principal Payment Dates

y and the bond is in premium ($P > \$100$).

So Treasury bonds are not at all riskfree, and its volatility is driven by the volatility of the interest rate. Assuming the high credit quality of the US government, the Treasury bonds are considered to be almost default free. During the heat of the debt-ceiling crisis in 2011, the rating agency S&P downgraded the US Treasury from AAA to AA+. The financial markets were in a crisis mode and Treasury bonds actually appreciated in value because, out of the flight to quality, investors move their capital away from risky assets to ... the US Treasury bonds.

The relation between P and y as expressed in Equation (1) is a very important one, and we will come back to it again. For now, I would like you to keep the picture of Figure 3 in mind. This is what the payoff schedule of a bond looks like. Over the life of the bond, you collect small coupon payments every six months, and toward the end of the life of the bond, at maturity, you collect the last coupon payment plus the principal. You discount this cashflow by a constant interest rate y using the discount function $1/(1 + y/2)^n$ for the n-th semi-annual payment. In doing this calculation, you link the bond price P to its yield to maturity y. There is no uncertainty involved in this relationship. There is also no economics involved in this calculation. But the calculation becomes very handy as we move between P and y. Concepts such as duration and convexity arise out of this calculation.

- **Treasury Yield Curve:** As shown in Figure 4, a Treasury yield curve is plot of yield against maturity, for Treasury bonds of varying maturities. Treasury bonds are traded in terms of market prices P. So a yield curve is constructed using the market prices of individual Treasury bonds. In Figure 4, the green dots are Treasury bills, the blue dots are Treasury notes, and the purple dots are old Treasury bonds. For example, the yield curve in Figure 4 was plotted for November 8, 1994. For a purple dot with a maturity of seven years, the bond was issued 23 years ago in 1971 as a 30-year Treasury bond.

As you can see, the yield curve is not created in vacuum. It is made up of individual bonds. In fact, the creation of a yield curve is not a simple task. The various bonds have different liquidity: the old bonds are typically less liquid while the new bonds/notes are typically very liquid. The liquidity effect shows up in the market prices of these bonds: illiquid bonds are cheaper than the liquid bonds. As a result, in constructing the yield curve, considerations such as liquidity take place. I do not want to make you a specialist in curve fitting, but if we have time in the next class, I will talk more about curve fitting.

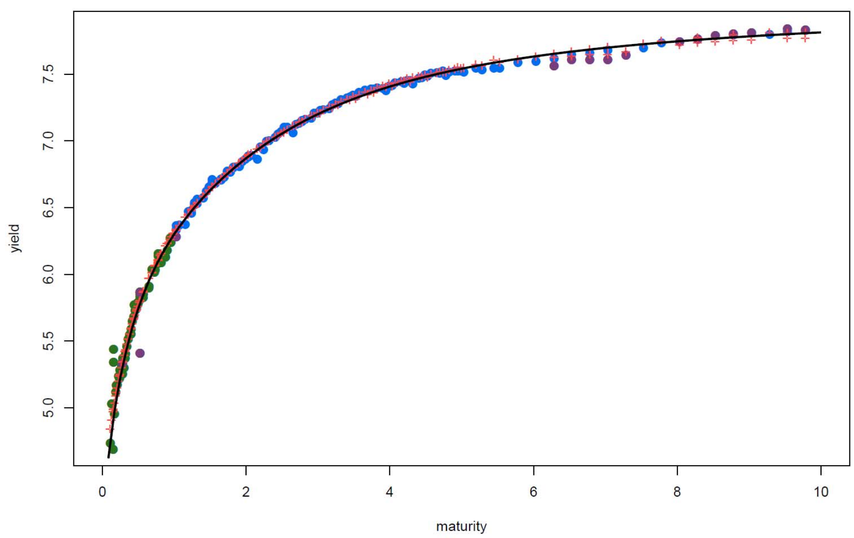

Figure 4: Treasury Yield Curve on November 8, 1994.

Focusing back on the yield curve in Figure 4, we see that on this day, the term structure is upward sloping. The short end of the yield curve is about 4.6%, the 2-year yield is about 6.8%, and the 10-year yield is at 7.8%. This makes the 10y to 2y spread at about 100 basis points. For bonds of similar maturities, the spreads are quite tight, indicating active arbitrage activities on the yield curve. By comparison, the yield curve on December 11, 2008, plotted in Figure 17, looks quite dramatic. Bonds are very similar maturities are trading at a yield spread in the order of 50 basis points. During normal market conditions, spreads so wide would never happen in this market. Of course, December 2008 was not normal. This picture indicates the lack of arbitrage activities in 2008, even in the most liquid market.

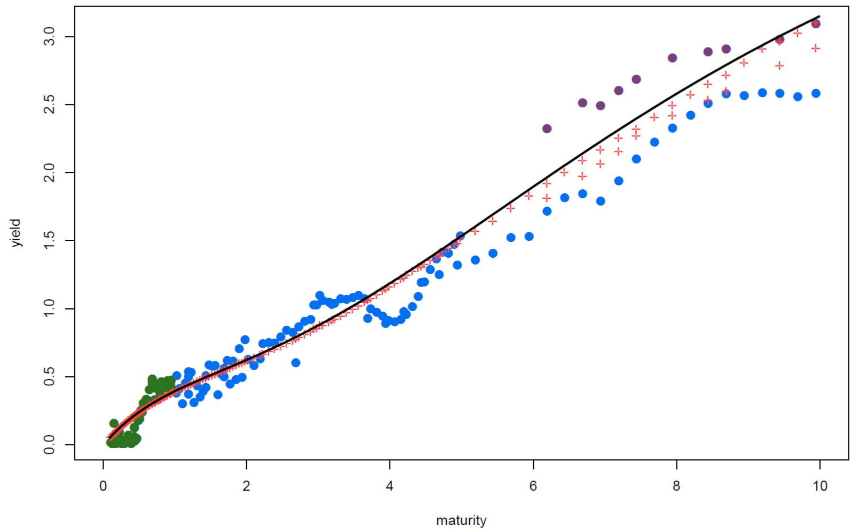

Figure 5: Treasury Yield Curve on December 11, 2008.

• **Time-Varying Yields:** To understand how the yield curve move over time, Figure 6 plots the time-series of Treasury constant maturity yields for a few selected maturities.

These constant maturity yields are calculated daily by using market prices of Treasury bonds as the input. And the output is the par-coupon yields of varying maturities. Effectively, these are interpolated yields for the a set of fixed maturity of interest (e.g., 1, 2, 3, 5, 7, 10, 20, and 30 years). Again, to know what is really going on, we need to spend some time on curve fitting. For those who are interested, this is a not so useful explanation from the Treasury department, but it is better than nothing.

Let's now used these CMT yields and see how the yield curve vary over time. As shown in Figure 6, most of the time, the yield curve is upward sloping. Using data from 1982 to today, the 2-year CMT yield is on average 4.97%, the 10-year yield is on average 6.09%, and the 30-year yield is on average 6.72%. So the spread of 10y to 2y is on average 100 basis points. There are also times when the yield curve is not so steep or even inverting. We will take a closer look later on these events. Also notice that the green line (2yr yield) is picking up in recent days. The 2yr yield is a policy sensitive yield and is moving up in anticipation of a rate hike.

Also notice the missing 30yr yield in Figure 6 from early 2002 to early 2006. In late 2001, facing projections of burgeoning surpluses, the Treasury decided to stop issuing the 30-year bond to save tax payers money. In late 2005, the Treasury decided to re-introduce the 30-year bond and held its first auction in fives years on February 9, 2006.

Using these CMT yields, let's also calculate the daily volatility of the Treasury yields. As

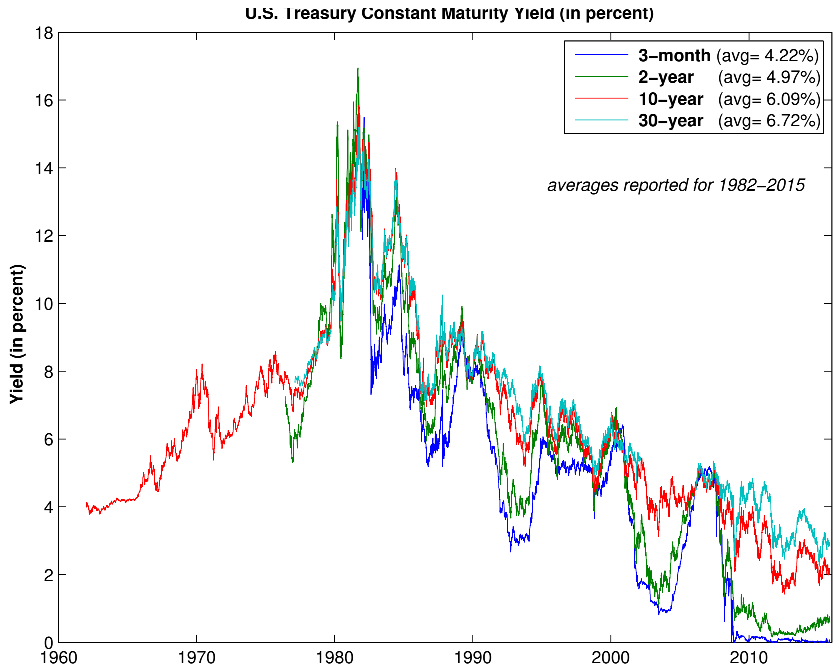

Figure 6: Time-Series of Treasury Constant Maturity Yields.

shown in Table 2, using daily data from 1982 to today, the standard deviation of the daily changes in the 3M Tbill rate is about 7.63 basis points. The 2Y and 10Y yields are slightly less volatile, at around 6.8 basis points. In recent years, however, the volatility is low for the short end because of the monetary policy. In general, however, the short end of the yield is typically more volatile, although the different in volatility is not huge. In other words, when measured in the yield space, the volatility across different maturity is comparable. But when it comes to the return space, the volatility across different maturity will be very different because of the difference in duration, which we will see shortly.

Table 2 also reports the largest one day movements for these yields. Let me link a few of these extreme movements in yield to the events at the time:

- October 20, 1987 was the day after the 1987 stock market crash.
- April 1994 was a very testy time in the bond market because of monetary policy tightening by Chairman Greenspan.
- September 15 to 19, 2008 was the week of Lehman default and AIG bailout. TBill rates first decreased sharply (increased in value) because of flight to quality and then bounced back on September 19.
- On March 18, 2009, the Fed made the following announcements, which were summarized in Chairman Ben Bernanke's recent book. *The overall package was designed to get*

Table 2: Summary Statistics of Daily Changes in Treasury Yields

sample	maturity	std (bp)	min (bp)	date	max (bp)	date
1982-2015	3M	7.63	-104	19820222	169	19820201
	2Y	6.86	-84	19871020	80	19820201
	10Y	6.80	-75	19871020	44	19820201
	30Y	6.30	-76	19871020	42	19820201
1990-2008	3M	5.18	-64	20070820	58	20001226
	2Y	6.05	-54	20010913	36	19940404
	10Y	5.78	-23	19950613	39	19940404
	30Y	4.99	-33	20011031	32	19940404
2008-2015	3M	4.94	-81	20080917	76	20080919
	2Y	4.86	-45	20080915	38	20080919
	10Y	6.42	-51	20090318	24	20080930
	30Y	6.12	-32	20081120	28	20110811

markets' attention, and it did. We announced that we planned to increase our 2009 purchases of mortgage-backed securities guaranteed by Fannie, Freddie, and Ginnie Mae to $1.25 trillion, an increase of $750 billion. We also doubled, from $100 billion to $200 billion, our planned purchases of the debt issued by Fannie and Freddie to finance their own holdings. We would also buy $300 billion of Treasuries over the next six months, our first foray into Treasury purchases. Finally, we strengthened our guidance about our plans for our benchmark interest rate, the federal funds rate. In January, we had said that we expected the funds rate to be at exceptionally low levels "for some time." In March, "for some time" became "for an extended period." We hoped that this new signal on short-term rates would help bring down long-term rates.

– The across-the-board increase in yield on February 1, 1982 was likely caused by the monetary policy tightening under Chairman Paul Volcker.

Overall, the numbers presented in Table 2 give us a baseline in observing and judging the daily movements in interest rates. A one-sigma move in this market is about 6 to 7 basis points. A daily movement of 25 basis points is unusual for this market.

- **Dollar Duration:** There are two measures of duration that is important for us to know. The dollar duration is defined as

$$-\frac{\partial P}{\partial y} = \frac{1}{1+\frac{y}{2}} \left[\sum_{n=1}^{2T} \frac{n}{2} \times \frac{\frac{c}{2}\times 100}{\left(1+\frac{y}{2}\right)^n} + T \times \frac{100}{\left(1+\frac{y}{2}\right)^{2T}} \right], \tag{2}$$

which is the negative of dollar change in bond price per unit change in yield. Given that a typical change in yield is measured in basis points, the often used DV01 measure scales the dollar measure by 10,000:

$$\text{DV01} = \text{Dollar Duration}/10,000\,,$$

which measures the negative change in bond price per one basis point change in yield.

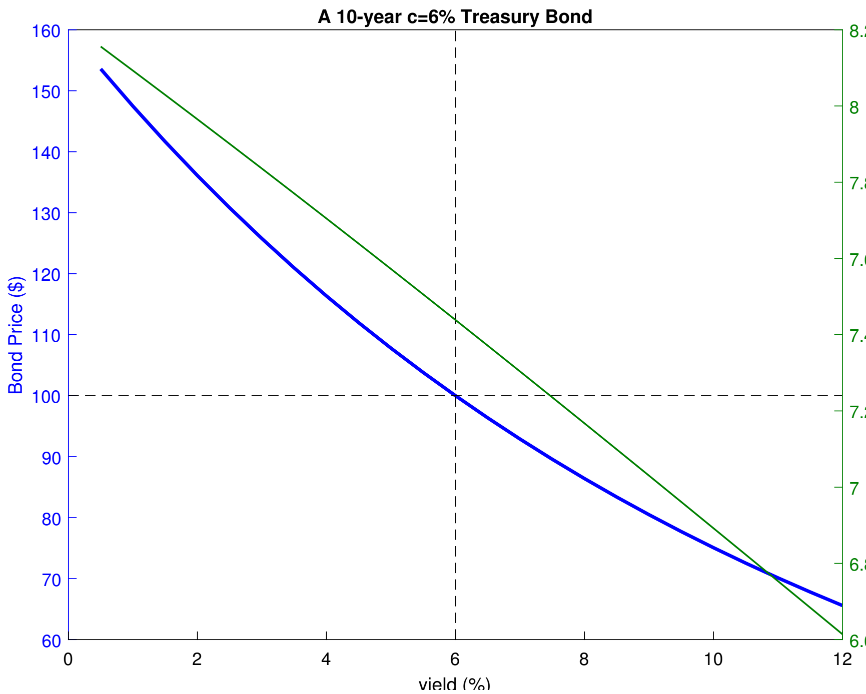

Figure 7: Bond Price as a Function of Yield and Duration as a Function of Yield

Figure 7 plots the bond price P as a function of yield y for a ten-year bond with coupon rate of 6%. Effectively, it plots the relation between P and y in Equation (1). As we can see, P is inversely related to y: decreasing y is coupled with increasing P. Also, the relation is not linear. But if we would like to approximate the relation linearly, we can pick a level of y, say $y = 6\%$ and $P = \$100$ and draw a tangent line at that point. As you've been taught many times in the past, the slope is $\partial P/\partial y$ as calculated in Equation (2). In other words, the dollar duration is the negative of the slope.

So if I would like to know how much I will lose when the ten-year Treasury yield suddenly increases by 10 basis points, I can use the linear approximation:

$$\Delta P_t = P_t - P_{t-1} \approx -\text{D}^{\$} \times (y_t - y_{t-1}) = -\text{D}^{\$} \times \Delta y_t = -D^{\$} \times \frac{10}{10,000} = -\text{DV01} \times 10\,\text{bps}$$

Going back to Figure 7, let's still focus just on the blue line. We notice that when y decreases, the slope gets steeper; when y increases, the slope gets flatter. This is because the relation between P and y as defined by Equation (1) is convex. For an investor holding a long position in bond, he would very much welcome this feature: profits due to decreasing y are amplified and losses due to increasing y are dampened.

- **Modified Duration:** The modified duration is defined as

$$-\frac{1}{P}\frac{\partial P}{\partial y} = \frac{1}{1+\frac{y}{2}} \frac{\sum_{n=1}^{2T} \frac{n}{2} \times \frac{\frac{c}{2}\times 100}{\left(1+\frac{y}{2}\right)^n} + T \times \frac{100}{\left(1+\frac{y}{2}\right)^{2T}}}{\sum_{n=1}^{2T} \frac{\frac{c}{2}\times 100}{\left(1+\frac{y}{2}\right)^n} + \frac{100}{\left(1+\frac{y}{2}\right)^{2T}}} \tag{3}$$

It is the dollar duration divided by the bond price. So its focus is on the profit/loss as a fraction of the position:

$$R_t = \frac{\Delta P_t}{P_{t-1}} = \frac{P_t - P_{t-1}}{P_{t-1}} - \approx \mathrm{D}^{\mathrm{mod}} \times (y_t - y_{t-1}) = -\mathrm{D}^{\mathrm{mod}} \times \Delta y_t$$

Dollar durations and modified durations are used for different purposes. If we are interested in the profit/loss in dollar terms, we go with the dollar duration, but if we interested in the profit/loss in the return space, we go with the modified duration.

As shown in Equation (3), the modified duration is a normalized measure and the unit is in year. In dealing with coupon bonds, it is always useful to go to the extreme and think first in terms of zero-coupon bond. For a T-year zero-coupon bond, the modified duration is T divided by $(1 + y/2)$. If instead of semi-annual compounding, the yield y is continuously compounded, then the modified duration of a T-year zero-coupon bond is simply T.

For a bond with semi-annual coupon payments, the modified duration is a weighted sum of all of the coupon payment dates, 0.5, 1.0, 1.5, ..., and T years. Except for the final date T, the n-th coupon dates are weighted by $\frac{c/2\times 100}{(1+y/2)^n}$. The last date T carries a disproportionately high weight because of the principal payment \$100. Because of this, the weighting is always tilted toward the final date T. To be more precise, date T is weighted by $\frac{c/2\times 100+100}{(1+y/2)^{2T}}$. For a coupon rate of 6%, $c/2 \times 100 + 100$ is 103, easily overpowering $c/2 = 3$.

You might wonder what happens when we have a really aggressive discount rate y, say $y = 10\%$? Well, let's consider the two extreme points: $\frac{1}{(1+y/2)^n}$ for the first coupon payment $n = 1$ and $\frac{1}{(1+y/2)^{2T}}$ for the final date T. Plugging $y = 10\%$, we have $\frac{1}{(1+y/2)} = 0.9524$ and $\frac{1}{(1+y/2)^{2T}} = 0.3769$ for $T = 10$. As you can see, even with this very aggressive discount rate discounting over a 10-year period, the principal payment of \$100 still dominates the calculation.

This is why, as you can see in Table 3, the modified duration of a ten-year bond is close to 10, especially when y is low. As y gets higher, this discounting effect becomes relatively

Table 3: Modified Duration

yield y	2%	3%	4%	5%	6%	6%	6%	7%	10%
coupon c	2%	3%	4%	5%	4.8%	6%	7.2%	7%	10%
$T = 1$	0.99	0.98	0.97	0.96	0.96	0.96	0.95	0.95	0.93
$T = 2$	1.95	1.93	1.90	1.88	1.87	1.86	1.84	1.84	1.77
$T = 3$	2.90	2.85	2.80	2.75	2.74	2.71	2.68	2.66	2.54
$T = 5$	4.74	4.61	4.49	4.38	4.36	4.27	4.18	4.16	3.86
$T = 7$	6.50	6.27	6.05	5.85	5.81	5.65	5.51	5.46	4.95
$T = 10$	9.02	8.58	8.18	7.79	7.71	7.44	7.21	7.11	6.23
$T = 20$	16.42	14.96	13.68	12.55	12.12	11.56	11.13	10.68	8.58
$T = 30$	22.48	19.69	17.38	15.45	14.46	13.84	13.39	12.47	9.46

more important, pushing the "center of gravity" away from T. As a result, the modified duration gets smaller.

Building on this analogy of "center of gravity" a little bit more, let's go back to the picture in Figure 3, which is a useful picture to have in our head when doing bond math. At least this is how I do the math. I imagine that there is a center of gravity along the horizontal dimension. Its gets pulled/pushed left and right, depending on the relative weights between the last date T and the other coupon dates. Getting pushed to the left results in a smaller duration and getting pull to the right results in a larger duration.

For example, consider two bonds with the same y and same T but different coupon rate c. It could be that one bond was issued back in 1990 as a 30-year bond and has five year to maturity. The other bond is a newly issued 5-year notes. Assuming a flat term structure of interest rate, the yields of these two bonds are the same, but their coupon rates are different (so are their bond prices). Which one has a higher duration? The one with lower c has its center of gravity closer to T. As a result, it has a higher duration.

Generally, it is useful to have a table like that in Table 3 handy, or build a function in Excel to calculate the modified duration of a bond for give coupon c, yield y, and maturity T. Historically, the average 10-year yield is about 6%. It is useful to know that, for a 10-year par coupon bond with $c = 6\%$, its modified duration is around 7.44 years. (Not precisely 7.44, but a number around 7 or 8.) In recent years, interest rates have been low, implying a relatively high duration for bonds. Right now (Fall 2015), the 10-year yield is at 2.34%. It would be useful to know that a 10-year par coupon bond with $c = 2\%$ has a modified duration around 9 years. The current 5-year yield is at 1.72%, and it is useful to know that a 5-year par coupon bond with $c = 2\%$ has a modified duration around 4.75 years. There is no need to memorize these numbers, but to have a rough sense in terms of orders of magnitudes would be handy.

For example, we know that a typical one-day one-sigma move in 10-year yield is about 6.8 basis points. How much does that translate to return volatility? Recall that, $R_t \approx \mathrm{D}^{\mathrm{mod}} \times \Delta y_t$. So, $\mathrm{std}(R_t) \approx \mathrm{D}^{\mathrm{mod}} \times std\,(\Delta y_t)$. For a 10-year bond with a duration of 7.44, a 6.8-bps volatility in Δy_t translates to $6.8 \times 7.44 = 50.6$ basis points in daily return volatility. Right now (Fall 2015), in a low interest rate environment, duration is high. For the same amount of volatility in Δy_t, the bond return volatility would be higher because of the higher duration.

As another example, suppose you believe that the 30-year bond is priced cheap relative to the yield curve. Your model tells you that the spread between the 30-year bond and the curve (generated by your model) is about 10 basis points. You believe that this spread is due to temporary illiquidity in 30-year bonds and will converge to close to zero later on. How much does this 10 basis points translate to return? Right now (Fall 2015), the 30-year yield is at 3.12%. Table 3 tells us that at this rate, the modified duration is about 20 years. So $R_t \approx -\mathrm{D}^{\mathrm{mod}} \times \Delta y_t = -20 \times (-10\,\mathrm{bps}) = 2\%$.

- **Duration and Convexity:** Concepts such as duration and convexity are only meaning because we work in the yield space and the profit/loss is in the dollar space. As such, duration serves as a bridge that connects the bond price to yield:

  – Dollar Duration:

$$\Delta P_t = P_t - P_{t-1} \approx -\mathrm{D}^{\$} \times (y_t - y_{t-1}) = -\mathrm{D}^{\$} \times \Delta y_t$$

  – Modified Duration:

$$R_t = \frac{\Delta P_t}{P_{t-1}} = \frac{P_t - P_{t-1}}{P_{t-1}} - \approx \mathrm{D}^{\mathrm{mod}} \times (y_t - y_{t-1}) = -\mathrm{D}^{\mathrm{mod}} \times \Delta y_t$$

In addition to this linear approximation through duration, we also notice that the relation between price and yield is not linear but convex. So convexity is introduced as a second-order approximation to improve upon the first order, linear approximation. In this class, we will not go for the exact formula for this second order approximation. If one day, you become a bond trader/portfolio manager, than you might be busy with convexity hedging. Even then, you might notice that the term structure of interest rate is not flat, which could cause quite a bit of problem for your first order duration hedge.

Let me close by talking about one intuition associated with convexity that is important. The relation between duration and yield is as plotted in Figure 7. With decreasing y, duration increases. As a result, the profit from holding a bond gets amplified. This effect is not symmetric in losses because with increasing y, duration decreases. As a result, the loss associated with holding a bond gets dampened. This positive convexity makes bond more attractive than a security that is linear in y. Later on, we will see a fixed-income security (Mortgage-Backed Securities) with negative convexity and use bonds (with positive convexity) to do duration hedge.

3 The Universe of Fixed Income Securities

Fixed-income securities share one thing: exposures to the Treasury yield curve. Most of these securities have an added component of credit risk. Muni's are bonds issued by municipalities, whose default probability is higher than the US government. The recent bankruptcy of Detroit is one example. Corporate bonds are issued by individual corporations, which also include credit risk. Agency bonds are issued by the government sponsored agencies (GSE) like Fannie and Freddie. After the government takeover in 2008, these bonds are explicitly backed by the US government. Prior to the takeover, it was implicitly backed by the government. For most of the fixed-income securities, the Treasury yield curve serves as a benchmark. Credit-sensitive instruments such as corporate bonds are usually quoted in terms of its spread relative to the US treasury yield.

Table 4 gives a summary of the US bond market. It gives us a sense of the relative size of the various components of the fixed-income market. In later classes, we will study the corporate bond market and will also touch upon the mortgage backed securities.

4 Factors Influencing the Yield Curve

• **The Yield Curve:** The Treasury yield curve is the best way to summarize the market prices of Treasury bonds, just like the implied-vol curves in the options market. In options, the vol curves become a three-dimensional surface because of an option could vary in its moneyness as well as time to expiration. In bonds, the yield curve remains a two-dimensional curve: a plot of yield against maturity.

Of course, bonds of the same maturity also vary in their "moneyness": new bonds are issued at par with $y = c$ and $P = \$100$; old bonds issued during high interest-rate environment are premium bonds with $c > y$ and $P > \$100$; bonds issued during extremely low interest-rate environment will eventually become discount bonds with $c < y$ and $P < \$100$. Because of this, when we talk about yield curve, we need to be more specific. In general, for coupon bonds, we usually use the par curve: the yields for par coupon bonds. For a given maturity, the yield of a par coupon bond will be located ... exactly on the curve, while the yields for

Table 4: Outstanding US Bond Market Debt in $ Billions

	Muni	Treasury	Mortgage Related	Corp Debt	Agency Bonds	Money Markets	Asset Backed	Total
1980	399.4	623.2	111.4	458.6	164.3	480.7		2,237.7
1981	443.7	720.3	127.0	489.2	194.5	593.7		2,568.4
1982	508.0	881.5	177.1	534.7	208.8	622.7		2,932.8
1983	575.1	1,050.9	248.3	575.3	209.3	638.3		3,297.2
1984	650.6	1,247.4	302.9	651.9	240.4	777.1		3,870.4
1985	859.5	1,437.7	399.9	776.6	261.0	950.9	1.2	4,686.7
1986	920.4	1,619.0	614.7	959.3	276.6	998.6	11.3	5,399.8
1987	1,012.0	1,724.7	816.0	1,074.9	308.3	1,125.8	18.1	6,079.7
1988	1,080.0	1,821.3	973.6	1,195.8	370.7	1,263.0	25.8	6,730.1
1989	1,129.8	1,945.4	1,192.7	1,292.4	397.5	1,359.5	37.3	7,354.6
1990	1,178.6	2,195.8	1,340.1	1,350.3	421.5	1,328.9	66.2	7,881.5
1991	1,272.1	2,471.6	1,577.1	1,454.6	421.5	1,215.7	91.7	8,504.3
1992	1,295.4	2,754.1	1,774.3	1,557.1	462.4	1,157.9	116.4	9,117.6
1993	1,361.7	2,989.5	2,209.0	1,782.8	550.8	1,143.6	132.5	10,170.0
1994	1,325.8	3,126.0	2,352.9	1,931.1	727.7	1,229.1	161.9	10,854.5
1995	1,268.2	3,307.2	2,432.1	2,087.5	924.0	1,367.6	214.9	11,601.4
1996	1,261.6	3,459.7	2,606.4	2,247.9	925.8	1,572.7	296.8	12,371.0
1997	1,318.5	3,456.8	2,871.8	2,457.5	1,021.8	1,871.1	392.5	13,390.0
1998	1,402.7	3,355.5	3,243.4	2,779.4	1,302.1	2,091.9	477.8	14,652.8
1999	1,457.1	3,266.0	3,832.2	3,120.0	1,620.0	2,452.7	583.5	16,331.5
2000	1,480.7	2,951.9	4,119.3	3,400.5	1,853.7	2,815.8	699.5	17,321.5
2001	1,603.4	2,967.5	4,711.0	3,824.6	2,157.4	2,715.0	811.9	18,790.8
2002	1,762.8	3,204.9	5,286.3	4,035.5	2,377.7	2,637.2	902.0	20,206.3
2003	1,900.4	3,574.9	5,708.0	4,310.4	2,626.2	2,616.1	992.7	21,728.6
2004	2,821.2	3,943.6	6,289.1	4,537.9	2,700.6	2,996.1	1,096.6	24,385.1
2005	3,019.3	4,165.9	7,206.4	4,604.0	2,616.0	3,536.6	1,275.0	26,423.2
2006	3,189.3	4,322.9	8,376.0	4,842.5	2,634.0	4,140.0	1,642.7	29,147.3
2007	3,424.8	4,516.7	9,372.6	5,254.3	2,906.2	4,310.8	1,938.8	31,724.2
2008	3,517.2	5,783.6	9,457.6	5,417.5	3,210.6	3,939.3	1,799.3	33,125.2
2009	3,672.5	7,260.6	9,341.6	5,934.5	2,727.5	3,243.9	1,682.1	33,862.7
2010	3,772.1	8,853.0	9,221.4	6,543.4	2,538.8	2,980.8	1,476.3	35,385.9
2011	3,719.4	9,928.4	9,043.8	6,618.1	2,326.9	2,719.3	1,330.0	35,685.9
2012	3,714.4	11,046.1	8,814.9	7,049.6	2,095.8	2,612.3	1,253.6	36,586.7
2013	3,671.2	11,854.4	8,720.1	7,458.6	2,056.9	2,713.7	1,252.5	37,727.3
2014	3,652.4	12,504.8	8,729.4	7,846.2	2,028.7	2,903.3	1,336.5	39,001.3
2015Q1	3,694.0	12,630.2	8,688.9	7,965.1	1,975.6	2,879.2	1,361.3	39,194.4

discount/premium bonds will be close to the curve but slightly off.

With an upward sloping yield curve, the yield of a premium bond sits below the par curve while the yield of a discount bond sits above the par curve. This is, because the premium bond, with relatively higher coupon payments $c^{\text{premium}} > c^{\text{par}} > c^{\text{discount}}$, puts a relatively higher weight on the yields of shorter maturities. With an upward sloping yield curve, this translates to a slightly lower yield. Overall, however, the differences are not huge. I would encourage you to go through the math yourself to verify this intuition and gauge the magnitude.

In doing these calculations, there is always a curve in the background that guides our intuition. That is the zero curve, which is effectively the collection of discount functions over different maturities. Having this zero curve is useful in discounting cashflows and we can price bonds of all maturities and varying coupon rates, using the discount function dictated by the zero curve. As a result, this zero curve enforces the pricing of bonds of all maturities to be internally consistency. Now the question is where do we get a zero curve in the first place? Most of the bonds in the market are coupon-bearing bonds, we do not observe the zero curve directly. So the common practice is to build a zero curve from market prices of coupon-bearing bonds. In fact, this task of yield curve fitting should be a very basic skill for a fixed-income person. If I have time for the next class (on term structure modeling), I'll talk more about the exact approach. By the way, using the intuitive developed earlier about premium/discount bonds, we know that, with an upward sloping yield curve, the zero curve sits above the par curve.

• **Factors Influencing the Yield Curve:** Our discussion so far focuses on the internal consistency of bond pricing. We imagine that there is a zero curve and ask the pricing of all coupon-bearing bonds to be consistent with this zero curve. The first question you would ask is: what are the factors influencing this zero curve? In an environment of constant interest rate, this zero-curve would always be flat. Then there is not too much to talk about. In practice, the curve is not flat and interest rates are not constant. So what can we learn about them? We will try to answer this question in today's class.

Once we are happy with the answers to the first question, we will ask the second question, which is also very interesting. With a zero curve (or even a sophisticated term structure model), we price all coupon-bearing bonds traded in the market. How good is our curve (or model) in pricing all of these bonds? What are we to learn when some bonds are mis-priced by a curve (or model)? We will try to answer this question in our next class.

As you can see, both questions focus on the same issue: what are the factors influencing bond pricing in the market place? I split the question into two so that we can answer this important question in two steps. First, we address the economic factors influencing the yield curve. In a way, these factors are more macro and systematic, affecting every "body" on the yield curve. Second, we address the institutional reasons affecting the yield curve. These factors are more localized and idiosyncratic.

• **Movements and Co-Movements in Yield Curve:** Figure 16 plots the time-series of Treasury yields of the representative maturities: 3M, 2Y, 5Y, 10Y, and 30Y. As you can see, most of the time, the yield curve is upward sloping. In a few occasion, the yield curve becomes flat or even inverts to a downward sloping curve.

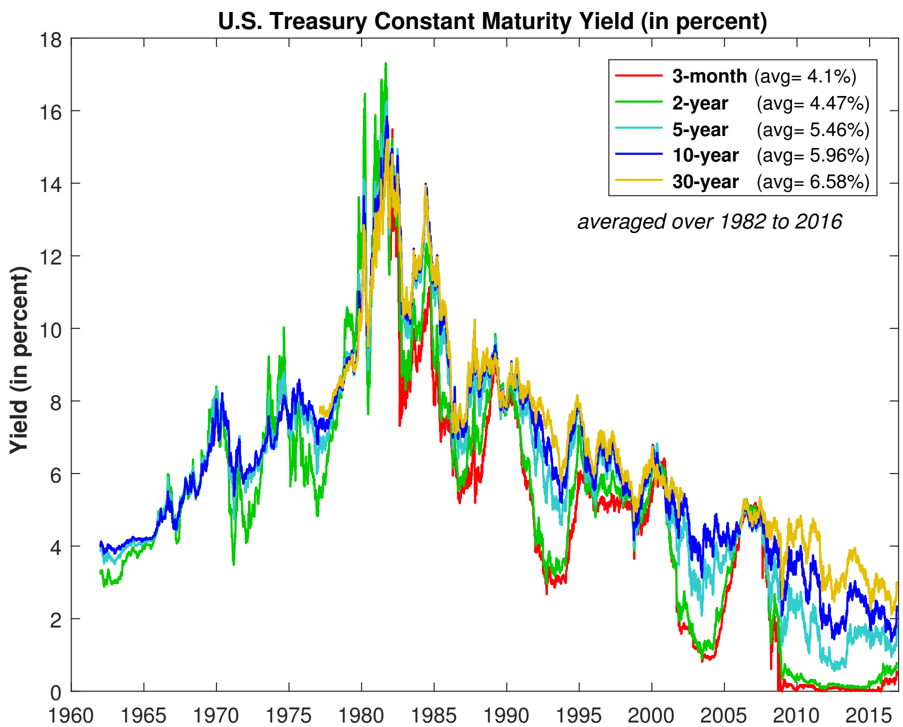

Figure 8: Time-Series of Treasury Constant Maturity Yields.

It is also evident from the plot that there is quite a bit of comovement in yields across different maturities. This is better summarized in Table 5. The pairwise correlations between yields of different maturities are well above 90%. Given the visible time trend and the persistence in yield (the auto-correlation in yield is close to 1), it is more meaningful to measure the comovement in changes in yields. After all, it is the surprise components (i.e., the random shocks) in yield that interest us the most. Measuring the pairwise correlations between daily changes in yields, we still find substantial comovements. Within the Treasury bonds and notes, the correlations between the two nearest maturities are above 90%. The comovement

Table 5: Comovement in Yields, Daily Data from 1982 to 2015

corr in yields (%)					
	3M	2Y	5Y	10Y	30Y
3M	100.0	98.57	96.19	93.61	90.90
2Y	**98.57**	100.0	99.18	97.54	95.47
5Y	96.19	**99.18**	100.0	99.46	98.19
10Y	93.61	97.54	**99.46**	100.0	99.57
30Y	90.90	95.47	98.19	**99.57**	100.0
corr in daily changes in yields (%)					
	3M	2Y	5Y	10Y	30Y
3M	100.0	57.31	46.87	40.18	35.15
2Y	**57.31**	100.0	90.29	82.17	72.90
5Y	46.87	**90.29**	100.0	94.07	85.74
10Y	40.18	82.17	**94.07**	100.0	93.71
30Y	35.15	72.90	85.74	**93.71**	100.0

becomes relatively weaker as the maturities are further apart. But even for the 2Y and 30Y bonds, the correlation is around 70%. The connection between the Treasury bills and the rest of the yield curve is relatively weaker but still substantial: the correlation between 3M TBill and 2Y bond is about 57%.

Overall, we can see that the Treasury yield curve is an inter-connected curve. It is not a curve with its individual components moving around freely without any regard for other parts of the curve. In this sense, the curve is a tight family of individual members. But it is also not a curve with its individual components moving in exactly the same pace. There is some internal consistency and relationship. The closer the maturity, the stronger the relationship. Let's try to figure out the economic factors that drive these movements and comovements.

• **Monetary Policy and Fed Funds Rate:** By far, the most important factor influencing the yield curve is monetary policy. In the US, monetary policy is carried out by the Federal Reserve through the Federal Open Market Committee (FOMC). In 1977, Congress set explicit objectives for monetary policy: "maximum employment" and "price stability." These two objectives in the Fed's so-called dual mandate are not always in alignment and the committee members of FOMC face the task of making the right decision when these two objectives are in conflict with each other.

The fed funds rate is the main policy tool of the Fed. It is the rate at which depository institutions lend excess reserve balances to each other overnight. Bank reserves are funds that banks hold at the Fed, much like the checking accounts that individuals have at banks. A bank can use its reserve account at the Fed for making or receiving payments from other banks, as well as a place to hold extra cash. Banks are legally required to hold a minimum level of reserves. If a bank finds itself with reserve balances in excess of the required minimum level, it often lends the excess reserves out to other banks in the so-called fed funds market, in the form of an unsecured overnight loan. And the interest rate of this private loan is the fed funds rate.

Quoting the former Chairman Ben Bernanke, "Although the federal funds rate is a private rate between banks, the Fed was able to control it indirectly by affecting the supply of funds available to banks. More precisely, the Fed managed the funds rate by affecting the quantity of bank reserves."

"The Fed was able to affect the quantity of bank reserves in the system, and thereby the federal funds rate, by buying or selling securities. When the Fed sells securities, for example, it gets paid by deducting their price from the reserve account of the purchaser's bank. The Fed's securities sales consequently drain reserves from the banking system. With fewer reserves available, banks are more eager to borrow from other banks, which puts upward pressure on the federal funds rate, the interest rate that banks pay on those borrowings. Similarly, to push down the federal funds rate, the Fed would buy securities, thereby adding to reserves in the banking system and reducing the need of banks to borrow from each other."

Effectively, the Fed's balance sheet is like a gigantic balloon attached to the entire US banking system. If the Fed feels that the economy is at the risk of overheating (i.e., the risk of high inflation), it will suck some air out of the system by selling securities into the system and therefore draining cash out of the banking system. If the Fed feels that the economy is performing poorly (i.e., the risk of high unemployment), it will blow some air into the system by buying securities from the system and therefore replenish the banking system with more cash.

In each of the FOMC meetings, the committee members weight the option of tightening (rate hike), loosening (rate cut), or no action. Over the history of FOMC meetings, the committee members are not always in agreement in terms of the right policy action. Hence the term *hawk*, who puts a higher weight on keeping the inflation low and often biases toward a tighter monetary policy; and *dove*, whose concern with respect to inflation is not as strong and often biases toward a loosening monetary policy. Of course, each policy decision is an "organic" process, with committee members taking into account of the information available to them at the time. If you read the memoirs of the former chairmen (e.g., Greenspan and Bernanke), you will notice that each decision weights heavily in their memory and on their conscience. Such men and women perform a great service to society.

- **Monetary Policy, Historical Experiences:** Figure 9 plots the time-series of Treasury yield curve (2-year and 10-year) along with the historical information that is relevant for our understanding of the monetary policy: inflation rate, GDP growth, and fed fund rates (the black solid line starting in the 1990s).

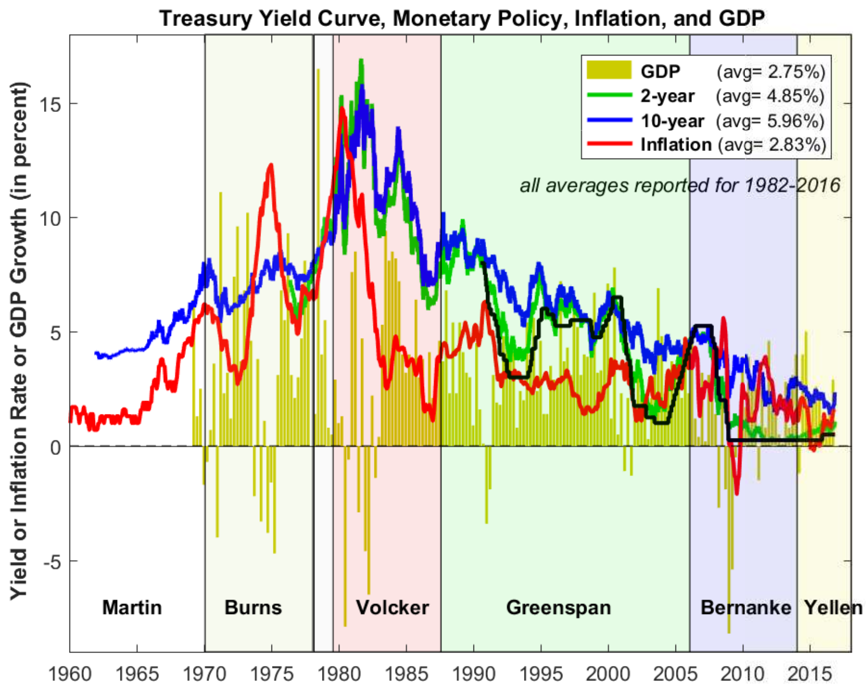

Figure 9: Treasury Yield Curve, Monetary Policy, Inflation Rate and GDP Growth.

Instead of writing about the historical experiences accompanying the events in Figure 9, let me use the following extended excerpt from Ben Bernanke's book. I find it to be very useful in my own understanding of the US monetary policy, especially for the pre-Greenspan era of Volcker and Burns. It was a period over which I know very little about because ... I was still in China and thought of interest rate as little rectangular stamps collected in a little booklet. Even that remotely related activity was performed only once or twice when my dad dragged a reluctant me, less than ten years of age, to a bank in an effort to educate me on the virtue of being frugal and the benefit of saving.

> Throughout most of the 1990s the Fed presided over an economy with employment growing strongly and inflation slowly declining to low levels. The Fed was thus meeting both parts of its congressional dual mandate to pursue maximum employment and price stability. In contrast, when I arrived at the Fed (August 2002), we saw risks to both sides of our mandate. On the employment side, we had the jobless recovery to contend with. On the price stability side, we faced a

problem unseen in the United States since the Depression – the possibility that inflation would fall too low or even tip into deflation, a broad decline in wages and prices.

In the past, the end of a recession had typically been followed by an improving jobs market. But during the two years after the recession that ended in November 2001, the U.S. economy actually lost 700,000 jobs, and unemployment edged up from 5.5 percent to 5.8 percent even as output grew. Many economists and pundits asked whether globalization and automation had somehow permanently damaged the U.S. economy's ability to create jobs. At the same time, inflation had been low and, with the economy sputtering, Fed economists warned that it could fall to 1/2 percent or below in 2003. Actual deflation could not be ruled out.

Worrying about possible deflation was a new experience for FOMC participants. Ever since the end of the Depression, the main risk to price stability had always been excessive inflation. Inflation spiraled up during the 1970s. Paul Volckers Fed ended it, but at a steep cost. Within a few months of Volckers becoming chairman in 1979, the Fed dramatically tightened monetary policy, and interest rates soared. By late 1981, the federal funds rate hit 20 percent and the interest rate on thirty-year fixed-rate mortgages topped 18 percent. As a consequence, housing, autos, and other credit-dependent industries screeched to a halt. A brief recession in 1980 was followed by a deep downturn in 1981-82. Unemployment crested above 10 percent, a rate last seen in the late 1930s.

After succeeding Volcker in 1987, Alan Greenspan continued the fight against inflation, although he was able to do so much more gradually and with fewer nasty side effects. By the late 1990s, the battle against high inflation appeared to be over. Inflation had fallen to about 2 percent per year, which seemed consistent with Greenspans informal definition of price stability: an inflation rate low enough that households and businesses did not take it into account when making economic decisions.

The Great Inflation of the 1970s had left a powerful impression on the minds of monetary policymakers. Michael Moskow, the president of the Federal Reserve Bank of Chicago when I joined the FOMC (August 2002), had served as an economist on the body that administered the infamous – and abjectly unsuccessful – Nixon wage-price controls, which had attempted to outlaw price increases. (Predictably, many suppliers managed to evade the controls, and, where they couldn't, some goods simply became unavailable when suppliers couldn't earn a profit selling at the mandated prices.) Don Kohn had been a Board staff economist in the

1970s under Fed chairman Arthur Burns, on whose watch inflation had surged. Greenspan himself had served as the chairman of President Ford's Council of Economic Advisers and no doubt shuddered to remember the Ford administrations ineffectual Whip Inflation Now campaign, which encouraged people to wear buttons signifying their commitment to taming the rising cost of living. With Fed policymakers conditioned to worry about too-high inflation, it was disorienting to consider that inflation might be too low. But it was a possibility that we would soon have to take seriously.

• **Fed Funds Rate and Yield Curve:** Beginning in 1994, the FOMC began announcing changes in its policy stance, and in 1995 it began to explicitly state its target level for the fed funds rate. As you can see in Figure 10, this aspect of monetary policy has an immediate impact on the Treasury yield curve, especially on the short end. In many instances, the bond market, in anticipation of the impending rate change, would price the event in advance. For example, on September 13, 2001, when the bond market re-opened on a limited basis after 9/11, the 3M Tbill rate dropped 52 bps from 3.26% to 2.74%, the one-year rate dropped 50 bps from 3.31% to 2.81%, and the two-year rate dropped 54 bps from 3.52% to 2.99%. It was not until four days later, on September 17, the Fed cut the fed fund target rate from 3.50% to 3%.

There is also a visible impact on the long-end of the yield curve, although the reactions of the longer end of the yield curve are not one-for-one in magnitude. This of course, makes sense given the impermanent nature of a monetary tightening or loosening. For example, On September 13, 2001, the five-year yield decreased by 38 bps from 4.41% to 4.03% and the ten-year yield decreased by 20 bps from 4.84% to 4.64%. Interestingly, the 30-year yield dropped by only 4 bps from 5.43% to 5.39%. (More on this topic on the 30-year yield next class.)

There were also times when the longer-term interest rates failed to rise after the Fed tightened monetary policy. This happened in 2004-05, the famous Greenspan's "conundrum." Quoting Bernanke again, "In speeches, I tied the conundrum to what I called the 'global savings glut' – more savings were available globally than there were good investments for those savings, and much of the excess foreign savings were flowing to the United States. Additional capital inflows resulted from efforts by (mostly) emerging-market countries like China to promote exports and reduce imports by keeping their currencies undervalued. To keep the value of its currency artificially low relative to the dollar, a country must stand ready to buy dollar-denominated assets, and China had purchased hundreds of billions of dollars' worth of U.S. debt, including mortgage-backed securities."

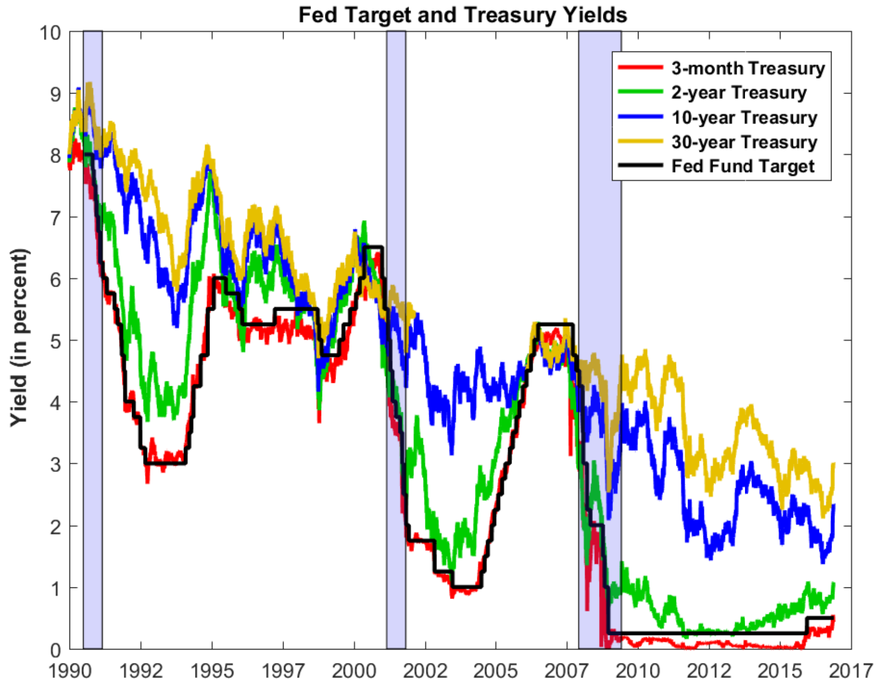

Figure 10: Treasury Yield Curve and Fed Fund Target Rate.

Before leaving Figure 10, let me point out one well-known pattern: Prior to each of the three NBER-dated recessions, the yield curve was either very flat or even inverted. It turns out that the slope of the interest rate is a good predictor for future GDP growth.

• **Factors Influencing Monetary Policy:** Also plotted in Figure 10 are the NBER-dated recession periods. The cyclical nature of the monetary policy is obvious in this plot: tightening during economic expansions and loosening during recessions. It is also worthwhile to note that, in order to cause minimal disruption to the markets, the monetary policy applies itself to the market gradually. A typical rate cut/hike is in increments of 25 bps. There were four rate hikes that were 50 bps (twice in 1994, once in 1995 and 2000) and one rate hike of 75 bps (November 1994). There were sixteen rate cuts of 50 bps (three times in 1991-92, nine times in 2001-02 and four times in 2007-08) and three rate cuts of 75 bps (all happened in 2008).

Given the dual mandate of price stability and maximum employment, it is not surprising that expectations of the rate of inflation, GDP growth, and employment numbers (e.g., nonfarm payroll employment) influence the decision of the policy decision of the FOMC. The Stanford economist John B. Taylor wrote a paper in 1993, linking the policy rate explicitly to inflation rate and GDP. This became the famous Taylor rule and there are various extensions of this rule. Again, if you read the memoirs of Greenspan and Bernanke, you would see that each policy decision is an "organic" process, with committee members taking into account of the information available to them at the time. Having a mechanic rule is useful as a baseline, but cannot be the ultimate answer.

If you pay attention to the famous Wall Street activity called the "Fed Watch," you will notice market participants use all kinds of signal trying to predict the next policy move. Some macro investors also perform directional trades to express their views and they typically do so using the two-year notes. As such, the two-year yield are considered to be highly sensitive to changes in the Fed's policy outlook. Consequently, the shape of the yield curve (relative to the two-year yield) might contain information about the impending policy move. As you can see in Figure 10, the two-year yield has been increasing quite steadily since the beginning of 2015 in anticipation of the monetary tightening in the end of December 2015.

In addition, investors also use fed funds futures traded on CME to express their views. Consequently, the pricing information in this market has been used to extract expectations of future Fed actions. This is a number watched closely by fixed-income traders and global macro investors. Even the Fed tracks this number to gauge the market expectation of their action. According to this calculation the implied probability of a rate hike from the current 25-50 bps to 50-75 bps is about 93.5%.

The market participants are involved in "Fed Watch" because uncertainties in the target rate have a big impact on the markets, not only the bond market but also the stock market (and the currency market). The Fed under chairman Bernanke and chairwoman Yellen has been working hard on Fed transparency in order to better communicate with the market participants in terms of the Fed policy.

• **Quantitative Easing and Operation Twist:** Earlier, we talked about how the Fed can use this gigantic balloon to suck/blow air into the entire banking system by selling/buying securities. Up to the 2008 crisis, the Fed performed monetary policy through affecting the fed funds rate. Starting from late 2008, the Fed employed a policy tool that is highly unorthodox and controversial: purchasing hundreds of billions of dollars of securities directly from the market with the intention to keep the long-term interest rates low.

After the FOMC meeting on October 29, 2008, the fed funds target rate was at 1% and the 3M Tbill rate was at 62 bps (the Treasury bill rates are lower than the overnight fed funds rate because of the potential counterparty risk involved in the unsecured fed funds loans). When the short-term interest rate reaches close to zero, what to do to bring down the longer-term interest rates in an effort to keep the economic recovery going? One way is to try to convince the market participants that the short-term interest rate will be kept low for a long time. In addition, the Fed also started to to purchase securities in an effort to directly influence the long-term interest rate. On November 25, 2008, the Fed announced plans to perform large scale asset purchases, often referred to as "Quantitative Easing" or QE. As shown in Figure 11, the actual purchases happened in December 2008 for agency bonds (Fannie and Freddie debt) and January 2009 for mortgage-backed securities backed by Fannie Mae, Freddie Mac, and Ginnie Mae. This was later known as QE1, because of it was followed by QE2 and QE3.

From Figure 11, you can see that the Fed also purchased around \$300 billion in Treasury securities during QE1, partly to supplement the reduction in MBS holdings when the mortgages underlying the MBS were paid off (either because of home sales or refinancings due the decreasing interest rate). We will visit this issue of negative convexity of MBS in a later class.

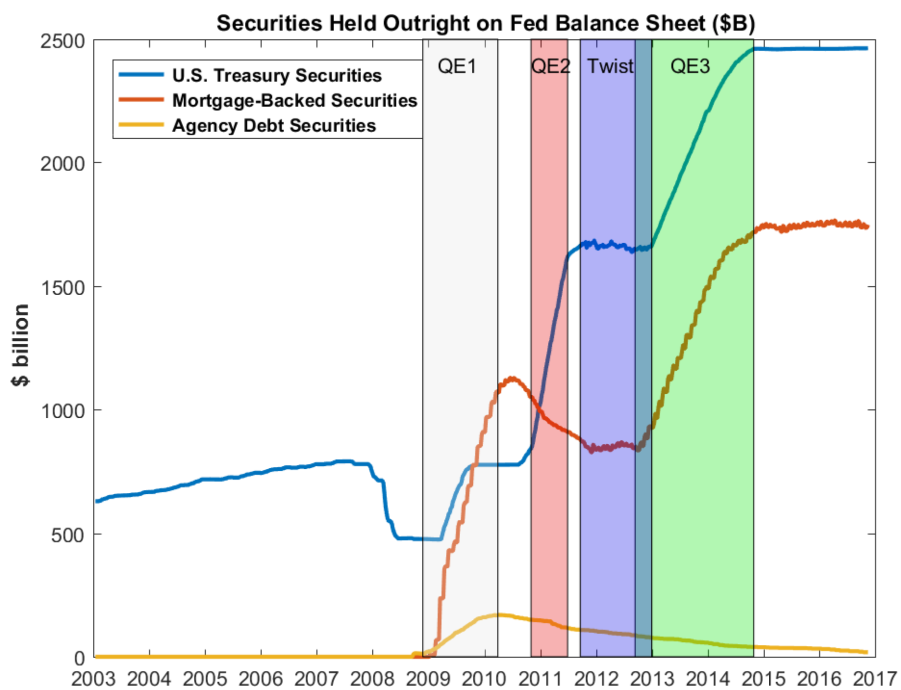

Figure 11: The Fed's Balance Sheet.

QE1 was followed by QE2 and QE3 and a program called "operation twist" in between. The securities purchased through QE2 and QE3 can be seen through the Fed's balance sheet in Figure 11. By now (November 11, 2015), the total market value of securities held outright on Fed's balance sheet is \$4.24 trillion, with \$2.46 trillion in US Treasury securities. To put these numbers in perspective, let's take a look at some other numbers. According to this Treasury website, as of August 2015, foreign holdings of the Treasury securities totals to $6.099 trillion with China holding $1.27 trillion and Japan holding $1.197 trillion. According to the World Bank, the 2014 GDP is $17.419 trillion for the US, $10.360 trillion for China, $4.601 trillion for Japan, $3.852 trillion for Germany, and $2.942 trillion for the UK.

Figure 12 plots the Fed's holdings of Treasury securities by maturity. As shown Figure 12, during "Operation Twist," the overall Treasury holding by the Fed remains nearly constant in market value. But the maturity of Fed's holdings went through a big change. As shown in Figure 12, the Fed was actively selling Treasuries securities maturing in 1-5 years and buying longer maturity bonds (5-10 years and longer than 10 years). Effectively, the Fed was increasing the duration of its Treasury portfolio without having to expand its balance sheet, in an effort to influence the long maturity yields so as to reduce the cost of credit for mortgage loans and corporate bonds.

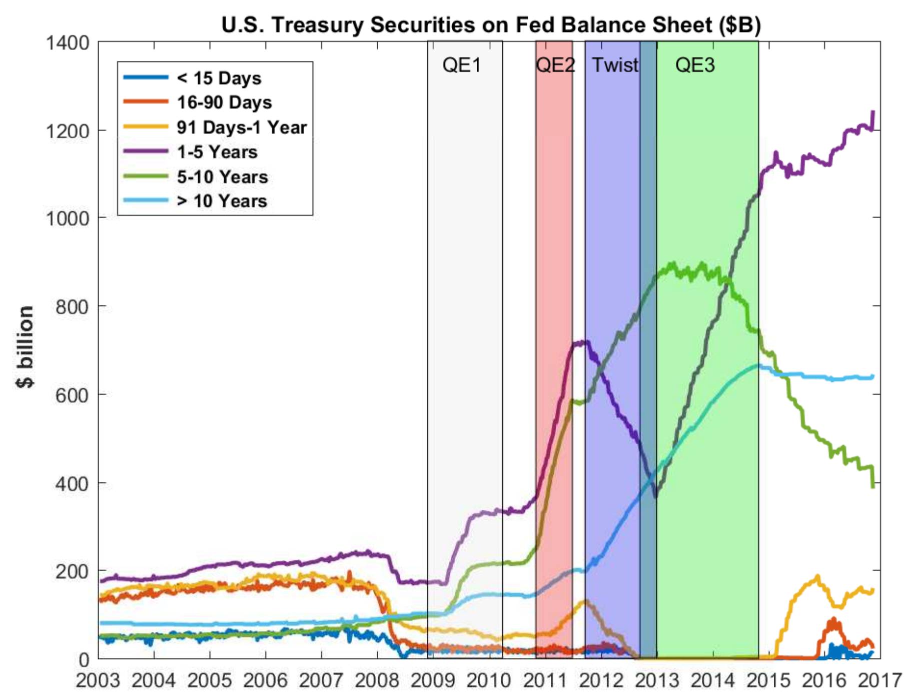

Figure 12: US Treasury Securities on Fed Balance Sheet, Maturity Composition.

The unconventional QE programs and the burgeoning Fed balance sheet were certainly not without risk. To put the policy thinking in perspective, let's take a look at the macro variables from 2008 to 2015 (see Figure 13). Prior to QE2, around October 2010, the unemployment rate was at 9.6%. The last time the unemployment rate was this high was during 1982-83 after the monetary tightening by Chairman Volcker. By contrast, the inflation was low at 1.1% in October 2010. Prior QE3, around August 2012, the employment rate was at 8.1% and the inflation was at 2%. It was clear that at the time, the Fed felt that the unemployment rates were too high (and inflation was not an issue of big concerns) and the economy needed help ... from somewhere. And the Fed's decision at the time was to step up and provide that help.

The economy has certainly been doing relatively better since then. As shown in Figure 13, by the end of QE3, the unemployment rate has been decreasing steadily to 5.7% and the GDP growth was at 4.3%. Right now (October 2015), the unemployment rate is at 5% and the GDP growth has been uneven: 1.5% for the third quarter and 3.9% for the second quarter. Overall, however, it is difficult to quantify the effect of the QE programs. How do you evaluate the counterfactual of an economy without QE programs? This, of course, is what differentiates Economics from Physics, where you can do controlled and repeated experiments.

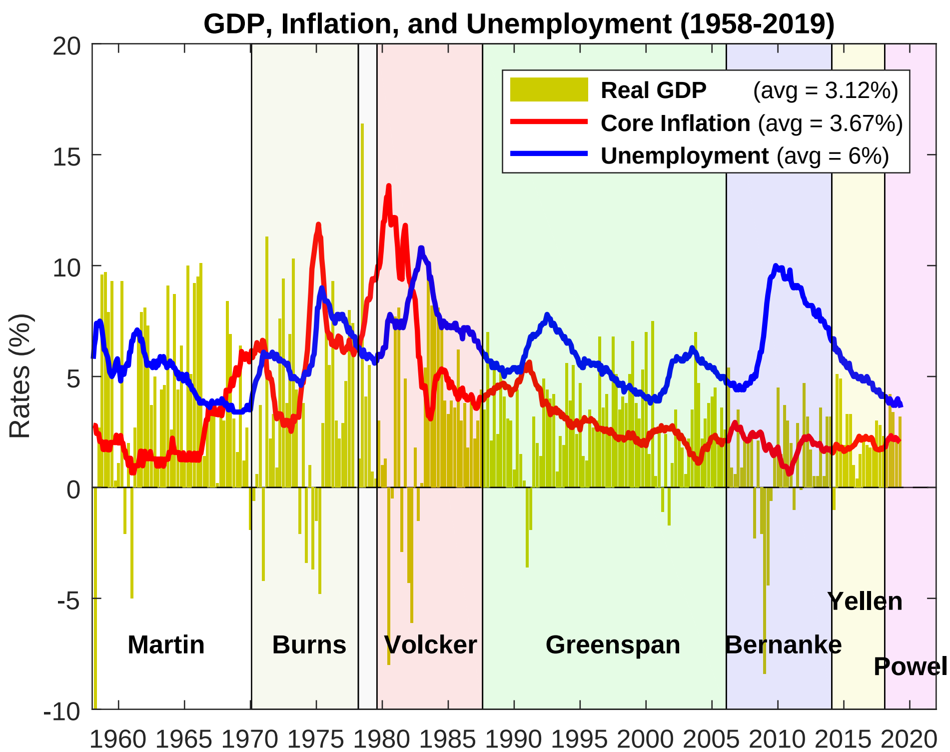

Figure 13: GDP, infation, and unemployment rate since 2008.

At the time, two major concerns about the QE programs were hyperinflation and sharp dollar depreciation. As shown in Figure 13, the inflation rate has in fact been unusually low in recent years with the 2015 numbers hovering around 0. If you follow the currency market, you would know that in recent years the dollar has been strengthening against most currencies, and this is true even before the recent election). So what were the reasons? Let me quote Ben Bernanke again:

> That idea (hyperinflation and sharp dollar depreciation) was linked to a perception that the Fed paid for securities by printing wheelbarrows of money. But contrary to what is sometimes said (and I said it once or twice myself, unfortunately, in oversimplified explanations), our policies did not involve printing money – neither literally, when referring to cash, nor even metaphorically, when referring to other forms of money such as checking accounts. The amount of currency in circulation is determined by how much cash people want to hold (the demand goes up around Christmas shopping time, for example) and is not affected by the Fed's securities purchases. Instead, the Fed pays for securities by creating reserves in the banking system. In a weak economy, like the one we were experiencing, those reserves simply lie fallow and they don't serve as 'money' in the common sense of the word.
>
> As the economy strengthened, banks would begin to loan out their reserves, which would ultimately lead to the expansion of money and credit. Up to a point, that was exactly what we wanted to see. If growth in money and credit became excessive, it would eventually result in inflation, but we could avoid that by unwinding our easy-money policies at the appropriate time. And, as I had explained on many occasions, we had the tools we needed to raise rates and tighten monetary policy when needed. The fears of hyperinflation or a collapse of the dollar were consequently quite exaggerated. Market indicators of inflation expectations – including the fact that the U.S. government was able to borrow long-term at very low interest rates – showed that investors had great confidence in the Fed's ability to keep inflation low. Our concern, if anything, was to get inflation a little higher, which was proving difficult to accomplish.

Finally, Figure 14 looks at the impact of the unconventional QE policy tools on the level as well as the slope of the yield curve. Again, causality is difficult to establish because we need to know the counterfactual of what would have happened if the Fed had not installed these policies. Also, the issue is further complicated by markets' anticipations at the time as well as the endogenuity of the decision itself. All in all, however, these policy actions seem to be effective in keeping the long-term interest rate low.

- **Why So Much on QEs?** If you feel that I am writing too much here on quantitative

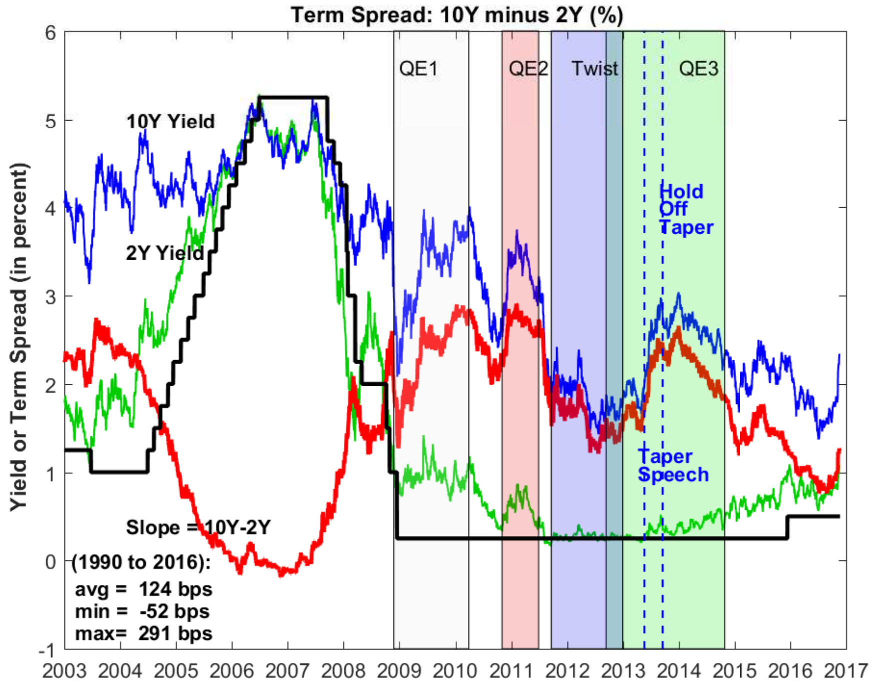

Figure 14: Treasury Yield Curve and Fed Fund Target Rate.

easings (more than you need to know), I agree. But more information is always better than no information, right? Rest assured, I'll not ask you to present the pro/con of the QE programs in the final exam.

I am recounting the events of 2010-2012 regarding quantitative easing for two reasons. First, these were really important events in the fixed income market. By going through the Fed's balance sheet, you get a better sense as to how the Fed's open market operation actually works. At least that was really helpful for me. The textbook information can be dry sometimes. Having plots like those in Figure 11 and 12 adds texture to my understanding. Second, I lived through that period in 2010-12 listening to many criticisms and derision against the QE programs. I am not a macro-economist and have not been trained in that field. And my thinking at the time was confused by many voices competing for attention. Personally, I find it is useful to read through the above two paragraphs written by Bernanke and look at the numbers for myself. So I thought I would share my readings with you.

I would not be surprised if, for each argument presented by Bernanke in his book, there is a counter argument. Honestly, the writings and thinking of some macro-economists are so complicated that they add more confusion than clarity. In my opinion, truth is always simple. It is the false that needs decoration. Complicated writing comes from a crowded and clouded mind. Unfortunately, in our field, complicated and convoluted thinking is often awarded with a premium because it is an exercise of a high IQ. In any case, for whatever it is worth, I appreciate the clear writing and thinking of chairman Bernanke.

5 Statistical Analysis of the Yield Curve

By now, we are comfortable with yield curves and have an intuitive understanding of the various factors influencing the short- and long-end of the yield curve. Let's now move on to quantify these random factors. Not surprisingly, the first risk factor that will show up through our analysis is the risk involved with duration. Second, we also noticed earlier that the entire yield curve does not move in tandem as in synchronized swimming. In particular, the long-end of the yield curve might not move entirely in parallel to the short-end of the yield curve. This points to the fact that the slope of the yield curve is not a constant. In fact, the slope becomes our second random factor. Finally, there might be some freedom in how the middle portion of the yield curve moves in relation to the short- and long-end. This observation gives rise to a third random factor called curvature.

It would not be surprising that market participants have long recognized the importance of these factors influencing the yield curve. But the concept of level, slope, and curvature was formally introduced in the 1991 paper by Litterman and Scheinkman, when both professors were working at Goldman Sachs. They identified these three common factors in the movements of yield curve through principal component analysis (PCA). In assignment 3, you get the chance to do this analysis yourself. The main difference is that their analysis is done in the yield space while your analysis will be done in the return space.

I'll go over this exercise in the yield space here in the notes.

• **Variance-Covariance Matrix:** Table 6 reproduces the content from Table 5 with the addition of 1Y yield. From Figure 16, we also notice that the 30Y yields were absent from February 19, 2002 to February 8, 2006 because the Treasury department suspended new issuance of 30-year bonds. In calculating the variance-covariance matrix, we will have to skip that specific period because of the missing 30-year bonds.

Let Cov be the variance-covariance matrix of the daily changes in yields for maturities 3M, 1Y, 2Y, 5Y, 10Y, and 30Y:

$$\mathrm{Cov}(i,j) = \mathrm{Corr}(i,j) \times \sigma_i \times \sigma_j \,,$$

where σ is the standard deviation of the daily changes in yield.

Table 6: Correlation and Standard Deviation of Daily Changes in Yields (1982 to 2015)

corr (%)	3M	1Y	2Y	5Y	10Y	30Y
3M	100.0	72.72	57.31	46.87	40.18	35.15
1Y	**72.24**	100.0	87.90	78.18	70.44	63.06
2Y	57.31	**87.90**	100.0	90.29	82.17	72.90
5Y	46.87	78.18	**90.29**	100.0	94.07	85.74
10Y	40.18	70.44	82.17	**94.07**	100.0	93.71
30Y	35.15	63.06	72.90	85.74	**93.71**	100.0
std (bps)	8.06	6.95	6.96	7.19	6.90	6.30

- **Eigenvalue Decomposition:** Taking Cov as an input, we perform the eigenvalue decomposition. Let's first go through the calculations and then come back to understand what is really going on. The eigenvalue decomposition will give us two inter-related outputs. First, the eigenvalue E is a vector of six eigenvalues. This is because the dimension of the variance-covariance matrix is 6, one for each maturity. As shown in Table 7, we order the eigenvalues in the order of their magnitude. We call the eigenvalue with the largest magnitude PC1 (principal component one), the second PC2, and so on. The magnitudes of the eigenvalues might not be meaningful for you now, but it will be.

Table 7: Eigenvalues and Eigenvectors

Eigenvalues E	PC1	PC2	PC3	PC4	PC5	PC6
E (bps^2)	226.99	50.14	13.77	5.45	2.86	1.47
E/sum(E) (%)	75.49	16.68	4.58	1.81	0.95	0.49

Eigenvectors D	PC1	PC2	PC3	PC4	PC5	PC6
3M	0.3630	-0.8017	0.4347	0.1876	-0.0365	0.0006
1Y	0.4182	-0.2371	-0.4682	-0.6806	0.2939	0.0016
2Y	0.4351	0.0257	-0.5134	0.3309	-0.6505	0.1176
5Y	0.4513	0.2493	-0.0709	0.4572	0.5076	-0.5124
10Y	0.4176	0.3430	0.2837	0.0418	0.2271	0.7577
30Y	0.3550	0.3472	0.4926	-0.4258	-0.4242	-0.3866

Second, associated with each eigenvalue is a vector, called eigenvector. There are six eigenvalues. So there are six eigenvectors, one for each eigenvalue. Putting these six eigenvectors together, we have a matrix D that is 6 by 6, as shown in Table 7. Let's now go over the first three PCs:

– **Level:** As shown in Table 7, associated with PC1 is the first eigenvector:

$$D^{PC1} = \begin{pmatrix} 0.3630 \\ 0.4182 \\ 0.4351 \\ 0.4513 \\ 0.4176 \\ 0.3550 \end{pmatrix},$$

which is a vector of six, one for each maturity. So effectively, the first PC is close to an equal-weighted portfolio of all six yields (or daily changes in yields, to be more precise). This factor corresponds to a movement in the yield curve when all six yields move up and down in tandem or in parallel. In other words, it captures the level movement and the best measure for exposure to this level risk is duration.

– **Slope:** Associated with PC2 is the second eigenvector:

$$D^{PC2} = \begin{pmatrix} -0.8017 \\ -0.2371 \\ 0.0257 \\ 0.2493 \\ 0.3430 \\ 0.3472 \end{pmatrix},$$

which is a long/short portfolio along the maturity dimension. It is long long-term yield and short short-term yield. It really does not matter which end of the yield curve is being long, as long as the weights on the long-end are opposite to the weights on the short-end. Naturally, you think "slope."

– **Curvature:** Associated with PC3 is the third eigenvector:

$$D^{PC3} = \begin{pmatrix} 0.4347 \\ -0.4682 \\ -0.5134 \\ -0.0709 \\ 0.2837 \\ 0.4926 \end{pmatrix},$$

which is again a long/short portfolio along the maturity dimension, but it is long both short- and long-end of the yield curve, and short the middle part of the yield curve.

Again, the exact sign of long/short does not really matter as long as the weights on the short- and long-end are opposite to the weights on the middle portion of the yield curve. So this reason, this factor is called "curvature."

Figure 15 summarizes the first three PCs in a plot, which might be more intuitive for us to see the meaning of level, slope, and curvature.

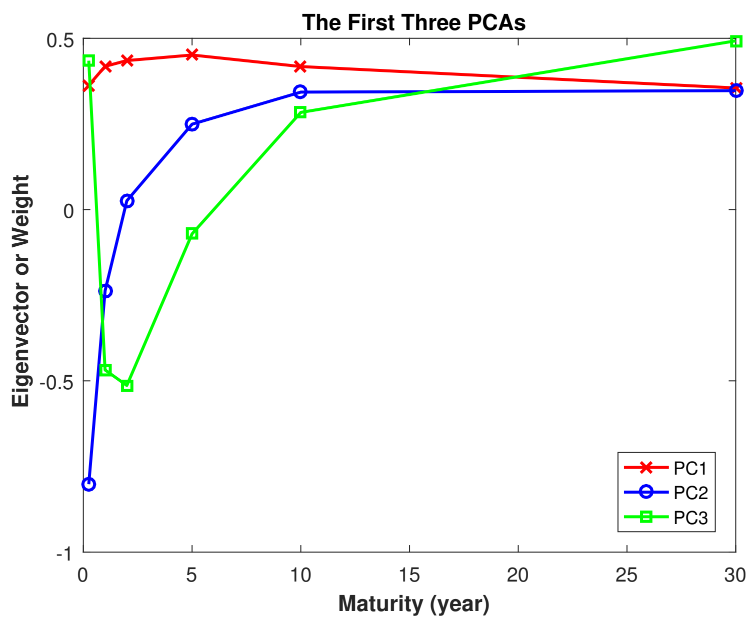

Figure 15: Level, Slope, and Curvature.

• **Relative Importance of the PCs:** We focus on the first three PCs because of their relative importance. To see this, let's go back to the eigenvalues in Table 7. By construction, the eigenvalue associated with PC1 is the highest in magnitude. Let's now construct a time-series of PC1 using the weights subscribed in D^{PC1} (avoiding the 2002-2006 period because of the missing 30-year yields). The standard deviation of this portfolio turns out to be 15.07 bps, and the variance is ... 226.98 (bps^2). You can repeat the same exercise for all other PCs. In short, the n-th eigenvalue is in fact the variance of the n-th PC.

What is cool about the eigenvalue decomposition is that it transforms the original data (with correlated yields) into six independent random factors: PC1, PC2, etc. (Please double check this statement by constructing time-series of PC1 and PC2 and calculate their correlation.) As a result, working with mutually independent PCs is more convenient than working with correlated yields. Because all six factors are independent, we can add now all six eigenvalues into sum(E) and use it as a normalizing factor for E. As shown in Table 7, the first PC accounts for 75.49% of the total variance, the second PC accounts for 16.68%, and the third PC accounts for 4.58%. Adding all three together, we see that they account for 96.75% of the total variance. This is why most of the term structure models use three factors. This is also why duration hedging, which is a hedge against PC1, is the most important form of hedging in the fixed income market.

Once a portfolio is hedged with zero duration, then the slope exposure becomes the most important risk. Once a portfolio is hedged with duration and slope, then you worry about curvature exposure. In the old days, there are butterfly trades which are duration and slope neutral, and are structured so that the main exposure is the curvature risk. Of course, you need to be a fixed-income nerd to get this deep into the yield curve trades.

• **More on the Eigenvectors D:** By now, we understand that there are six eigenvectors and putting them together gives us a 6×6 matrix. Each eigenvector is a vector of portfolio weights (not normalized) on the six maturities.

Let's now take a closer look. Let's start with the important observation that all six PCs are independent. So pick any two PCs, say PC1 and PC2, and their correlation will be zero. As mentioned earlier, this is why eigenvalue decomposition is useful. It gives us independent factors. Let's use the matrix notation for the following calculations. First we know that

$$PC1_t = \left(D^{PC1}\right)^\intercal \times \Delta\, y_t\,,$$

where Δy_t is the vector of daily changes in yields for the six maturities and $\left(D^{PC1}\right)^\intercal$ is the transpose of D^{PC1}. Of course, we also know that

$$PC2_t = \left(D^{PC2}\right)^\intercal \times \Delta\, y_t\,.$$

So if $\text{cov}\left(PC1_t, PC2_t\right) = 0$, then it must be that

$$\left(D^{PC1}\right)^\intercal \times D^{PC2} = 0\,.$$

Applying this logic pairwise to all maturities, you will be convinced that

$$D^\intercal\, D = I\,,$$

where I is an identity matrix of dimension 6×6, with diagonal terms equaling 1 and off-diagonal terms equaling zero. In other words,

$$D^{-1} = D^{\intercal}\,.$$

If you don't believe me, just try it out using Excel or Matlab.

• **Running Regressions:** Now let's take a look at Table 8, where I report the following regression results:

$$\Delta y_t = a + \beta^{PC1}\,PC1_t + \beta^{PC2}\,PC2_t + \beta^{PC3}\,PC3_t + \epsilon_t\,.$$

Knowing that all three PCs are independent, we can calculate the individual R-squared for each PC and add them together to get the total R-squared of the regression.

Table 8: Regressing Δy on the First Three PCs

maturity	PC1 β	PC2 β	PC3 β	PC1 R2 (%)	PC2 R2 (%)	PC3 R2 (%)	Total R2 (%)
3M	0.3630	-0.8017	0.4347	46.06	49.63	4.01	99.70
1Y	0.4182	-0.2371	-0.4682	82.18	5.83	6.25	94.26
2Y	0.4351	0.0257	-0.5134	88.67	0.07	7.49	96.23
5Y	0.4513	0.2493	-0.0709	89.46	6.03	0.13	95.62
10Y	0.4176	0.3430	0.2837	83.17	12.39	2.33	97.89
30Y	0.3550	0.3472	0.4926	72.04	15.22	8.41	95.66

First, you can see that PC1 remains the most important random factor, explaining the daily changes in yields with very high R-squared's. For the two extreme ends of the yield curve (3M and 30Y), the explanatory power is relatively weaker. This is where PC2 picks up. In particular, PC2 contributes quite a bit in explaining the movements in the short-end of the yield curve. Adding all three PC factors, we can explain the random variations in daily changes in yields with R-squared's that are well above 90%.

Second, take a look at the regression coefficients β's. What do you see? Compare D^{PC1} with the beta coefficients on PC1, there are identical! Likewise for D^{PC2} and β^{PC2}, and D^{PC3} and β^{PC3}. Can you prove this result mathematically? (No worries, I'll not ask you to do this proof in the exam.)

• **More on PCA:** What we've talked about so far in this section is statistical based. The yield curve is well suited for a statistical analysis like PCA. Once you understand the mechanics of the PCA, it will be instructive for you to go back to the economic drivers for these common risk factors in the fixed-income market.

More broadly, the PCA approach can also be used in many markets where the observables are correlated due to some common factors. For example, applying PCA to international equity returns, the first PC will be a world index with roughly equal weight on all countries. The second PC will be a long/short portfolio across the two most representative regions (which could change over time).

Whatever you might do with PCA, just be reminded that this is simply a statistical tool that helps you extract mutually independent factors, and the importance of the factors are ordered by their variances (i.e., eigenvalues). Also remember that the only input for the eigenvalue decomposition is the variance-covariance matrix. Use this tool effectively for the your desired objective. All all is done, take the extra step to understand the economic and institutional drivers for the extracted factors.

6 Term Structure Models

• **The Challenge from the Data:** In the fixed income market, term structure models are used to model interest rates. The challenge from the data has two dimensions. First, it should take into account of how the interest rates move over time. Second, for a given time, it should be able to model the yield curve, also called the term structure of interest rates. Figure 16 is a good summary of these two challenges from the data: a good term structure model should be able to capture the dynamic variations of the level of interest rates and the shape of the yield curve.

These two demands from the data are very similar to those in the equity market. A good model for stock market returns should be able to take into account of how stock returns vary over time, as well as how, for a give time, the cross-section of stocks are priced in relation to one another. In the equity market, an i.i.d. model for stock returns is a reasonable approximation. As such, the dynamics for stock returns are really simple: constant expected return μ, constant volatility σ, and unpredictable random shocks ϵ_{t+1}. Cross-sectionally, the expected stock returns are linked to one another through their exposures (i.e., betas) to risk factors in a model such as the CAPM. As such, the CAPM model is a static model with constant expected returns and constant beta.

The need for a dynamic model shows up when we investigated the time-varying volatility in our volatility class and stochastic volatility in our options class. Here in this class, we have a chance to take a closer look at these dynamic models.

• **Term Structure Models, Historical Development:** Term structure models were developed in the mid-1970s by Cox, Ingersoll and Ross (1985) and Vasicek (1997). You

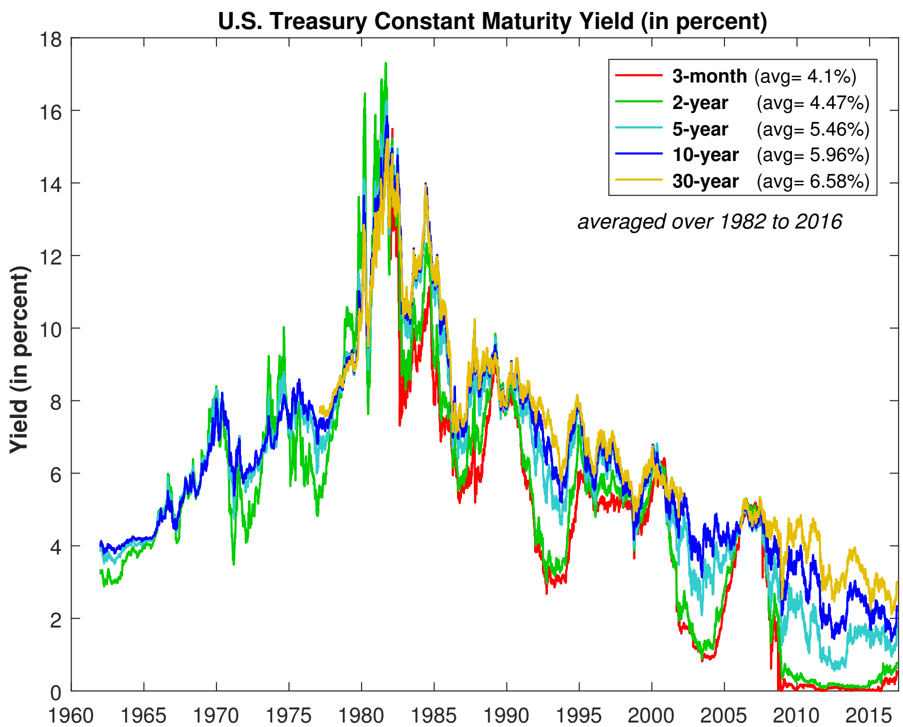

Figure 16: Time-Series of Treasury Constant Maturity Yields.

might notice that the CIR paper was published in 1985, but it was really a product of the mid-1970s. These term structure models were a continuation of the work done by Black, Merton, and Scholes, who popularized the application of continuous-time models in Finance. Like the Black-Scholes model before them, these term-structure models use the stochastic processes studied by mathematicians and physicists. For example, the CIR model builds on the Feller process and the Vasicek model builds on the Ornstein-Uhlenbeck process. In both cases, the starting point is the instantaneous short-rate r_t, which is modeled by a stochastic process (OU or Feller). The entire yield curve is priced using the dynamics of this one short rate. As such, the CIR and Vasicek models are one-factor short-rate models.

The second wave of term structure models came in the 1990s. When I entered the Stanford PhD program in 1995, I was just in time to catch the excitement surrounding term structure models. Relative to the original models of CIR and Vasicek, the effort of the new generation of term structure models is to be empirically relevant. From the work of Litterman and Scheinkman (1991), it became clear that a one-factor model will not be able to capture the entire shape of the yield curve. Unlike the stock market, where you can dismiss the risk uncaptured by the model as idiosyncratic risk, we do not have the luxury of dismissing common risk factors in the fixed income market (e.g., the slope factor).

These multifactor models quickly found their way into the "real" world. It is my understanding that each investment bank has its own proprietary term-structure model. And I was told by some practitioners that the industry has the best and most sophisticated term structure models. And they use these models to manage and hedge interest-rate risk (level, slope, convexity, volatility, etc) as well as to price interest-rate derivatives and other rate-sensitive instruments such as MBS. Looking back, I can now understand why during the mid-1990s, the Wall Street hired so many physicists and mathematicians. Most of my classmates in Physics ended up on Wall Street. I can also understand the sudden demand for more sophisticated term structure models in the 1990s. The fixed income desks were very profitable and the range and trading volume of their fixed income products were also expanding very rapidly during that time.

By now, the excitement surrounding term structure models has all but fizzled out. As a PhD student at Stanford, I spent much more time learning and working on term-structure models than anything else I did there. Since coming to Sloan in 2000, I have not made much use of that part of my training. Nevertheless, I am very grateful to my advisers at Stanford for having trained me in this area. As I wrote earlier in my lecture notes, not everything we do in life is of practical use. Still, they are useful and meaningful in our growth process.

For our class, however, I don't want to emphasize too much on the modeling part, because it takes quite a bit of mathematical skills. Instead, I would like to use the term structure models as a way for us to understand conceptually how the various parts of the yield curve are connected through a pricing model and the role of the risk factors in generating the pricing results.

- **Bond Pricing in Continuous-Time:** Let r_t be the time-t instantaneous short rate. Let today be time 0, and let P_0 be the present value of a dollar to be paid in T years. Discounting this future dollar all the way from T to today using the short rate, we have:

$$P_0 = E\left(e^{-\int_0^T r_t\,dt}\right) \tag{4}$$

Let me explain this expression in sequence:

  - The reason why we need to do $\int_0^T r_t\,dt$ is because we have to add up all of the future short rates along the path from 0 to T. Take the extreme example of a constant short rate r. We have $\int_0^T r_t\,dt = r\,T$ and $P_0 = e^{-r\,T}$.
  - We put $\int_0^T r_t\,dt$ onto $e^{-\int_0^T r_t\,dt}$ because the rates are continuously compounded. (You will find that working with e^x and $\ln(x)$ typically gives us a lot of tractability in Finance.)
  - Later on, we will see how r_t is going to be driven by a random risk factor. Because of this, there could be many paths of r_t, depending on the random outcomes of the risk factor. And the present value of a future dollar to be paid in year T is an expectation, $E(\cdot)$, taken over all potential random paths of r_t with t running from 0 to T.

- **Relating back to Option Pricing:** The calculation in Equation (4) is similar to the calculation of $E^Q\left(e^{-rT}\left(K - S_T\right)\mathbf{1}_{S_T<K}\right)$ in option pricing. The difference is that we do not have to deal with the random variation in S_T. But we have to deal with the random variation in the riskfree r, which turns out to be more difficult to deal with.

Instead of fixing a maturity date for this interest rate r (as in yields to maturity), we choose to work with the "short rate" so that this one rate can be used to discount future cashflows over any horizon. We just need to add them up via $\int_0^T r_t\,dt$.

A by-product of this modeling choice is that we now have to keep track of the entire path of r_t from 0 to T in order to calculate $\int_0^T r_t\,dt$. Remember that when you performed option pricing via simulation in your Assignment 3, you didn't have to keep track the path of S_T from 0 to T. You only needed to know the values of S_T. So in order to have one million scenarios of S_T, you needed to simulate one million random variables.

To price bonds, however, you need to simulate the entire path of r_t from 0 to T. Suppose we decide to discretize the time interval from 0 to T into monthly intervals, then pricing a one-year bond with one million scenarios would involve simulating 12 $\times$ one million random variables; pricing a 10-year bond would involve simulation 120 $\times$ one million random variables. In short, pricing bond is generally more involving than pricing equity options and pricing bond derivatives would be even more challenging. That is why models with closed-form solutions are very useful. Otherwise, we will have to resort to either simulations or solving partial differential equations.

Also notice that to be precise, I should take the expectation in Equation (4) under the risk-neutral measure. For this class, however, let me not make a distinction between the two, just to keep things simple.

- **The Vasicek Model:** In the Vasicek model, the short rate r_t follows

$$dr_t = \kappa \left(\bar{r} - r_t\right) \, dt + \sigma \, dB_t \, , \tag{5}$$

where, as in the Black-Scholes model, $\sigma \, dB_t$ is the diffusion component with B as a Brownian motion. This model has three parameters:

  - $\bar{r}$: The long-run mean of the interest rate, $\bar{r} = E(r_t)$.
  - κ: The rate of mean reversion. When r_t is above its long-run mean $\bar{r}$, $\bar{r} - r_t$ is negative, exerting a negative pull on r_t to make it closer to $\bar{r}$. A larger κ amplifies this pull of mean reversion and a smaller κ dampens it. Conversely, when r_t is below its long-run mean $\bar{r}$, $\bar{r} - r_t$ is positive, exerting a positive pull on r_t, again to make it closer to its long-run mean $\bar{r}$.
  - σ: controls the volatility of the interest rate.

- **Bond Pricing under Vasicek:** Bond pricing under the Vasicek model turns out of be very simple. Let today be time t and let r_t be today's short rate, then the time-t value of a dollar to be paid T years later at time $t + T$ is

$$P_t = e^{A + B \, r_t} \, ,$$

where

$$B = \frac{e^{-\kappa T} - 1}{\kappa}$$
$$A = \bar{r} \left(\frac{1 - e^{-\kappa T}}{\kappa} - T \right) + \frac{\sigma^2}{2\kappa^2} \left(\frac{1 - e^{-2\kappa T}}{2\kappa} - 2 \frac{1 - e^{-\kappa T}}{\kappa} + T \right)$$

7 Calibrating the Model to the Data

- **The Vasicek Model:** As usual, we work with models in order to understand, at a conceptual level, the key drivers in the pricing of a security. Applying the model to the data, we further understand quantitatively how well the model works and what's missing in the model.

  For a one-factor model such as the Vasicek model, we know its limitation even before applying it to the data. In the fixed income market, the level of the interest rates is the number one risk factor in terms of its importance, but it is not the only risk factor.

  In Assignment 4, I ask you to work with a discrete-time version of the Vasicek model by first estimating the model parameters, $\bar{r}$, κ, and σ, using the time-series data of 3-month Tbill rates. Basically, I am asking you to calibrate the model only to the time-series information of the short-end of the yield curve, without allowing you to take into account of the information contained in the other parts of the yield curve. Then I ask you to price the entire yield curve. Not surprising, you will find that the calibrated model does not work very well to accommodate the different shapes of the yield curve.

  An alternative approach is to calibrate the model using the yield curve. For example, on any given day, we estimate the model parameters, $\bar{r}$, κ, and σ, so that the pricing errors between the model yields and the market yields are minimized. By doing so, the model will do a much better job in matching the market observed yield curve, but it will miss the time-series information. Moreover, you will have one set of parameters per day, which is inconsistent with the assumption that these parameters are constant.

  The better solution is to introduce more factors to the model. For example, instead of forcing the long-run mean $\bar{r}$ to be a constant, we can allow it to vary over time by modeling it as a stochastic process. Instead of forcing the volatility coefficient σ to be a constant, we can allow it to vary as another stochastic process. There, you have a three-factor model. The pricing will be more complicated and so will be the estimation. Working with these multi-factor models requires some patience, perseverance, and the love for the subject matter. Indeed, it is not for everybody.

- **Curve Fitting:** On a topic related to model calibration is yield curve fitting. In this approach, there is no consideration along the time-series dimension. The zero rate $r(\tau)$ of maturity τ is modeled as a parametric function, which is then used to price all market traded coupon-bearing bonds. On any given day, the parameters in that parametric function will be chosen so that the pricing errors between the model yields and the market yields are minimized. This exercise of yield curve fitting is repeated daily and the model parameters are updated daily as well.

Figure 17 plots the yield curve during the depth of the 2008 crisis. It uses the Svensson model for curve fitting. The parameters in the Svensson model are first optimized so that the model can price all of the market-traded bonds on December 11, 2008 with minimum pricing errors. Using these parameters, the black line is the corresponding par coupon curve. The blue or purple dots are the market yields for the market-traded bonds. For each dot, there is a companion red "+", which is the model yield for the corresponding bond. In a fast decreasing interest rate environment such as December 2008, most of the existing bonds are premium bonds. As we discussed earlier, with an upward sloping term structure, the yields of these bonds are lower than the corresponding par-coupon yields of the same maturity. That is why most of the red "+"s are below the par coupon curve. If there are many discount bonds being traded at the time, then you will see some red "+"s above the par coupon curve.

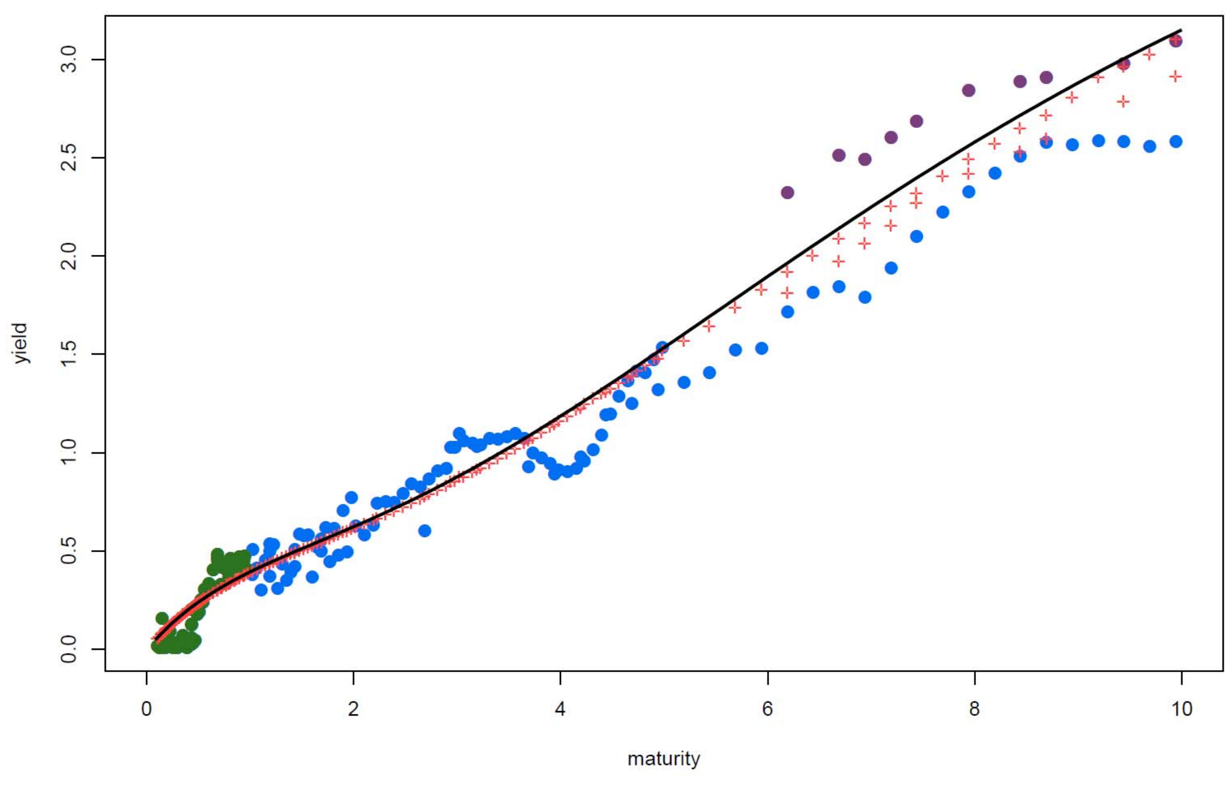

Figure 17: Treasury Yield Curve on December 11, 2008.

This curve fitting exercise is useful in connecting the yields of different maturities through a parametric function of zero rates. For example, there is quite a bit of overlap in discount rates between a ten-year yield and a ten-year minus one-month yield. The presence of a parametric function of zero rates acknowledges the overlap (ten years minus one month) and the pricing difference between these two yields will be sensitive only to the one-month gap. But the usefulness of a curve fitting exercise stops at this level. If you would like to use a model to help you with derivatives pricing on the yield curve (e.g., Bond options, swaptions, caps/floors, etc), a curve-fitting model will not be helpful at all because it does not take into consideration of how yields vary over time. For derivatives pricing on the yield curve, you need to use dynamic models. The usual approach is to use multi-factor versions of CIR or Vasicek models. Affine models are examples of these multi-factor versions of CIR and Vasicek.

8 Relative Value Trading with a Term Structure Model

In March 2011, Chifu Huang (a former MIT Sloan Finance professor) came to Prof. Merton's class to give a guest lecture. I found his talk to be very informative and the following is based on one portion of his talk.

• **How to Use a Term-Structure Model to Identify Trading Opportunity:** Relative value trading in the fixed income market does not make a judgment on the level of interest rates or the slope of the curve. It assumes that a few points on the yield curve are always fair. For example, the time-series data on the 10yr, 2yr, and 1-month rates can be used to estimate a three-factor term structure model.

Recall that in the Vasicek model, the short rate is the only risk factor (i.e., state variable). That is why in your Assignment 4, I ask you to estimate the model using only the 3M Tbill rates. With a three-factor model, we have three risk factors (i.e., state variables) and we need three points on the yield curve to help us estimate the model. Intuitively, the 10yr gives us information about the level of long-term interest rates; the 2yr together with the 10yr informs us about the slope of the curve; and the 1-month Tbill rate captures the short-term interest rate (including expectations on monetary policy in the near term).

Once you have the model estimated by the time-series data (which is a non-trivial task if you would like to do it properly), this model is going to give you predictions about the level of interest rates across the entire yield curve. You can then compare the model price with the market price to judge for yourself whether or not a market price is cheap or expensive. Once you convince yourself that your model helps you pick up a trading opportunity, you would structure a trade around it. You can buy cheap maturities and sell expensive maturities, and, at the same time, hedge your portfolio so that it is insensitive to the changes of the level or the slope of the yield curve.

The main judgment call is to understand why your model identifies some maturities as cheap or expensive. If it is due to institutional reasons (which does not show up in your model but does show up in the data), then you can make judgment as to whether or not such institutional reasons will dissipate over time (and how fast).

- **An Example:**

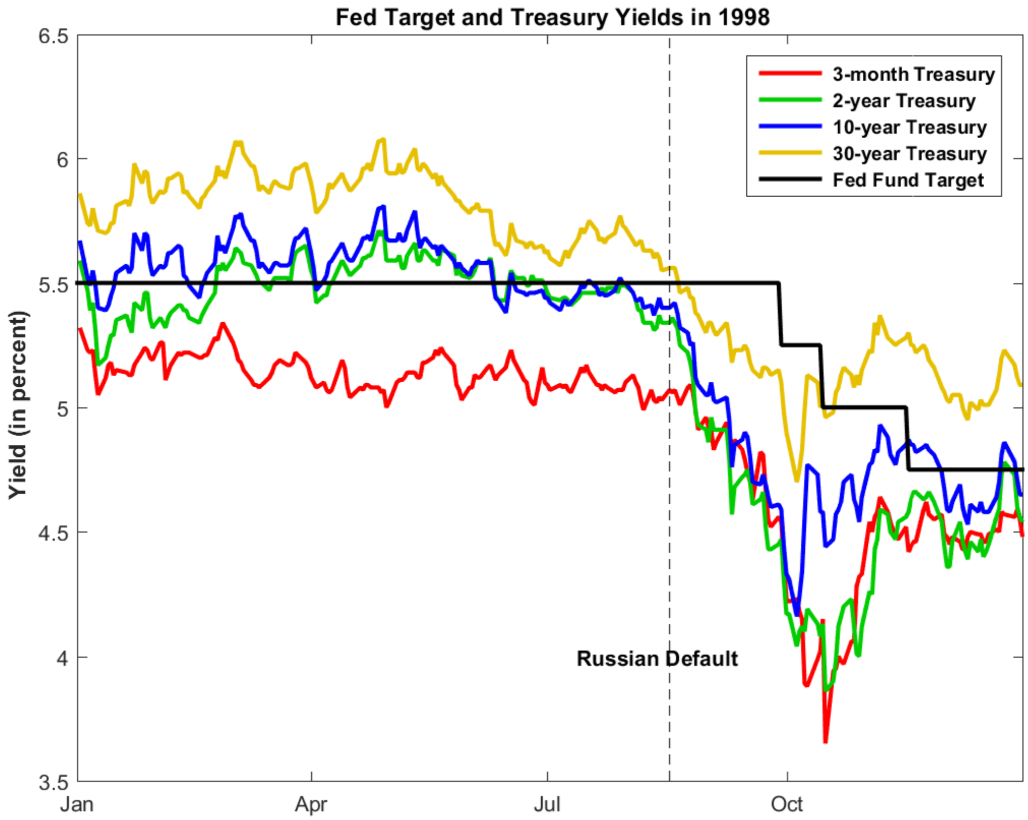

Figure 18: Fed Target and Treasury Yields in 1998.

One example was given by Chifu. In August 1998, Russian defaulted on its local currency debt, and the effect lingered well into September and was later known as the LTCM crisis. As shown in Figure 18, in September 1998, bond markets rallied in anticipation of a rate cut. On September 29, the Fed cut the fed funds target rate by 25 bps.

Figure 19 is a slide presented by Chifu in his talk. In September 1998, his two-factor model picks up a trading opportunity regarding the 30yr bond. According to the model, the market price for the 30yr bond is cheap relative to the model price. The deviation between the data and the model was at the range of 10 to 20 bps. The 30-year rate was around 5.5% at that time, implying a modified duration of about 15 years. So a 10 bps price deviation in 30yr would translate to 10 bps $\times$ 15 = 150 bps in bond return. And a 20 bps deviation will translate to 3% in bond return.

Figure 19: Cheapness and Richness of US 30-Year Swap Rate Based on a Two-Factor Model.

So what are the reasons for this cheapening of 30yr? It is because residing over the 30yr region are pension funds and life insurance companies who are either inactive "portfolio rebalancers" or rate-targeted buyers. As a result, the rally that happened in the rest of the yield curve didn't find its way to the 30yr region. There is a lag in how information (regarding an impending rate cut) gets transmitted to this region. As you can see from Figure 18 and 19, it was only after the Fed's rate cut on September 28 when the 30yr yield was brought back in alignment with the rest of the yield curve.