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Classes 14 & 15: Options, Part 2

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The Black-Scholes option pricing model, along with the arbitrage-free risk-neutral pricing framework, is something of a revolution in Finance. It managed to attract many mathematicians, physicists, and even engineers to Finance. But if the progression stopped right at the level of modeling and pricing, it would have been rather boring: you take the pricing formula, plug in the numbers, and get the price. So things would have been pretty mechanical. Real life is always more interesting than financial models. In this class, let's bring the model to the data and enjoy the discovery process.

1 Bring the Black-Scholes Model to the Data

- **ATM Options and Time-Varying Volatility:** Volatility plays a central role in option pricing. In the Black-Scholes model, volatility σ is a constant. If you take this assumption literally, then the Black-Scholes implied vol σ_t^I should be a constant over time. In practice, this is not at all true. As we learned in our earlier class on time-varying volatility, using either SMA or EWMA models, the volatility measured from the underlying stock market moves over time. Recall this plot, Figure 1, in Classes 8 & 9, where the option-implied volatility is plotted against the volatility measured directly from the underlying stock market. In both cases, stock return volatility varies over time.

  One interesting observation offered by Figure 1 is that the option-implied volatility is usually higher than the actual realized volatility in the stock market. In other words, within the Black-Scholes model, the options are more expensive than what can be justified by the underlying stock market volatility. If you believe in the Black-Scholes model, then selling volatility (via selling near-the-money options, calls or puts) will be a very profitable trading strategy.

  Figure 2 plots the time-series of VIX (option-implied volatility using SPX) against the time-series of the S&P 500 index level. As you can see, the random shocks to

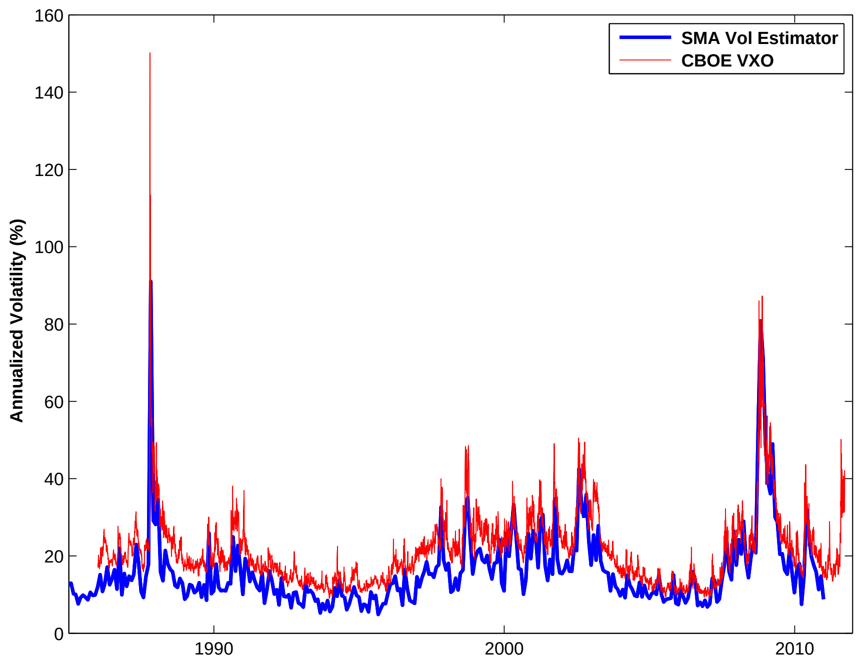

Figure 1: Time-Varying Volatility of the S&P 500 Index. The red line is the option-implied volatility using SPX traded on CBOE. The blue line is measured directly from the underlying stock market using daily returns of the S&P index.

VIX, especially those sudden increases in VIX are often accompanied by sudden and large drops in the index level. Of course, this observation is outside of the Black-Scholes model, where σ is a constant. But this plot gives us the intuition as to what could go wrong with selling volatility: you lose money when the markets are in crisis. Basically, by selling volatility on the overall market (e.g., SPX), your capital is at risk exactly when capital is scarce. In the language of the CAPM, you have a positive beta exposure.

But this positive beta exposure is more subtle than the simple linear co-movement captured by beta. As highlighted by the shaded areas, volatility typically spikes up when there are large crises. Just to name a few: the October 1987 stock market crash, the January 1991 Iraq war, the September 1993 Sterling crisis, the 1997 Asian crisis, the 1998 LTCM crisis, 9/11, 2002 Internet bubble burst, 2005 downgrade of GM and Ford, 2007 pre-crisis, March 2008 Bear Stearns, September 2008 Lehman, the European and Greek crises in 2010 and 2011, and the August 2015 Chinese spillover. In other words, what captured by Figure 2 is co-movement in extreme events, like the crisis beta in Assignment 1 (risk exposure conditioning on large negative stock returns). Also, as shown in Figure 2, not all crises have the same impact. For example, the downgrade of GM and Ford was a big event for the credit market, but not too scary for equity and index options.

The comovement in Figure 2 gives rise to a negative correlation between the S&P 500 index returns and changes in VIX, which ranges between -50% to -90%. Figure 3 is an old plot from Classes 8 & 9, which uses the EWMA model to estimate the correlation between the two. As you can see from the plot, the correlation has experienced a regime change. During the early sample period, the correlation hovers around -50%, while in more recent period, the correlation has become more severe, hovering around -80%.

All of these observations have direct impact on how options should be priced in practice: the Black-Scholes model need to allow σ to vary over time. The time variation of σ should not be modeled in a deterministic fashion. As shown in Figure 2, the time series of σ_t is affected by uncertain, random shocks. So just like the stock price S_t follows a stochastic process (e.g., geometric Brownian motion), σ_t itself should follow a stochastic process with its own random shocks. Moreover, the random shocks in σ_t should be negatively correlated with the random shocks in S_t to match the empirical evidence in Figure 2. There is a class of diffusion models called stochastic volatility models developed exactly for this purpose.

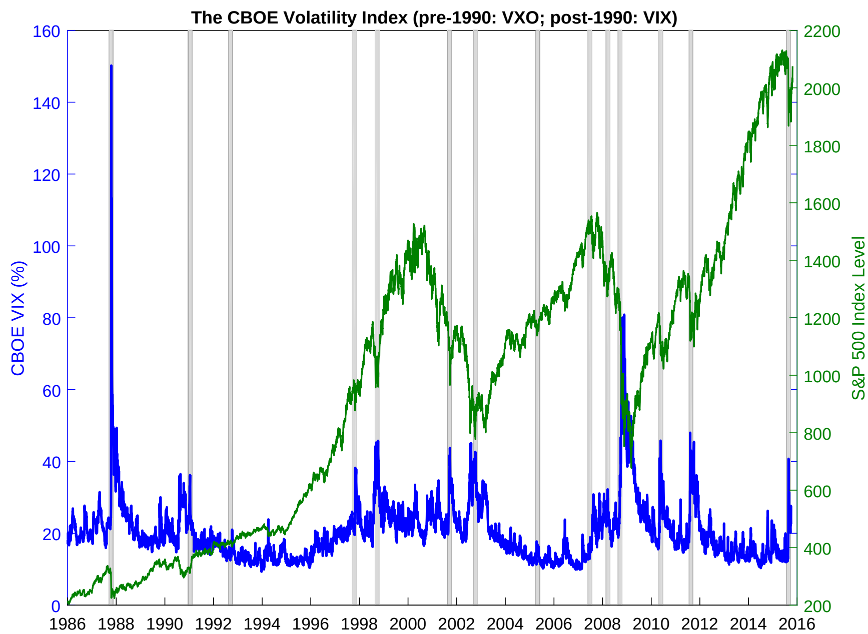

Figure 2: Time-Series of the CBOE VIX Index Plotted against the Time-Series of the S&P 500 Index Level. Prior to 1990, the old VIX (VXO) is used. Post 1990, the news VIX index is used.

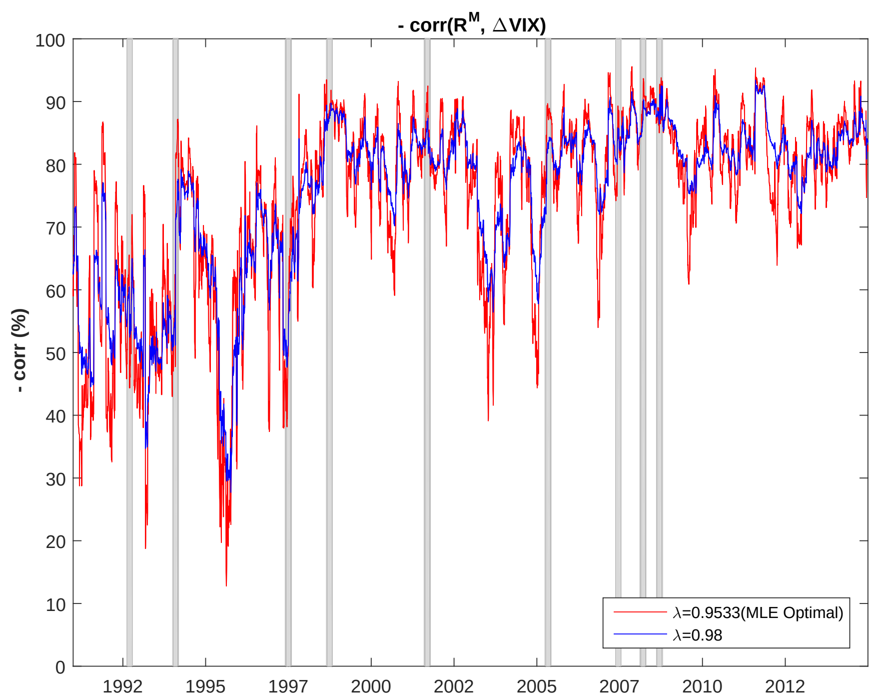

Figure 3: The Time-Series of EWMA Estimates for the Correlations between the S&P 500 Index Returns and Daily Changes in the VIX Index.

These models are similar to the discrete-time models like EWMA or GARCH, which also allow volatility to be time-varying. But one distinct future of stochastic volatility models is that it has its own random shocks. In EWMA or GARCH, the time-varying volatility comes from the random shocks in the stock market. We will come back to the stochastic volatility model later in the class, which are very useful in pricing options of different times to expiration, linking the pricing of long-dated options to that of short-dated options.

- **OTM Options and Tail Events:** In developing our intuition for the Black-Scholes model, we've focused mostly on the ATM options, which are important vehicles for volatility exposure. Now let's look at the pricing of the out-of-the-money options.

  Recall the risk-neutral pricing of a call option,

$$C_0 = E^Q\left(e^{-rT}(S_T - K)\mathbf{1}_{S_T>K}\right) = \boxed{e^{-rT}\, E^Q\left(S_T \mathbf{1}_{S_T>K}\right)} - \boxed{e^{-rT}\, K\, E^Q\left(\mathbf{1}_{S_T>K}\right)},$$

  where the pricing bolts down to calculations involving $E^Q(\mathbf{1}_{S_T>K}$ $E^Q(S_T\mathbf{1}_{S_T>K}$. For $K > S_0 e^{rT}$, the call option is out of the money. In fact, the larger the strike price K, the more out of the money the option is, and the smaller $E^Q(\mathbf{1}_{S_T>K}$. So if we focus on OTM calls, we zoom into the right tail.

  Likewise, the risk-neutral pricing of a put option is,

$$P_0 = E^Q\left(e^{-rT}(K - S_T)\mathbf{1}_{S_T<K}\right) = \boxed{e^{-rT}\, K\, E^Q\left(\mathbf{1}_{S_T<K}\right)} - \boxed{e^{-rT}\, E^Q\left(S_T \mathbf{1}_{S_T<K}\right)},$$

  where the pricing bolts down to calculations involving $E^Q(\mathbf{1}_{S_T<K}$ $E^Q(S_T\mathbf{1}_{S_T<K}$. For $K < S_0 e^{rT}$, the put option is out of the money. In fact, the smaller the strike price K, the more out of the money the option is, and the smaller $E^Q(\mathbf{1}_{S_T<K}$. So if we focus on OTM puts, we zoom into the left tail.

  Within the Black-Scholes model, the above calculations can be taken to the next level using the probability distribution of a standard normal:

$$P_0 = \boxed{e^{-rT}\, K\, E^Q\left(\mathbf{1}_{S_T<K}\right)} - \boxed{e^{-rT}\, E^Q\left(S_T \mathbf{1}_{S_T<K}\right)} = \boxed{e^{-rT}\, K\, N(-d_2)} - \boxed{S_0\, N(-d_1)},$$

  where I've changed the color coding so that this equation matches with Figure 4. More specifically, for a 10% OTM put striking at $K = S_0\, e^{r\,T} \times 90\%$, we can re-write the

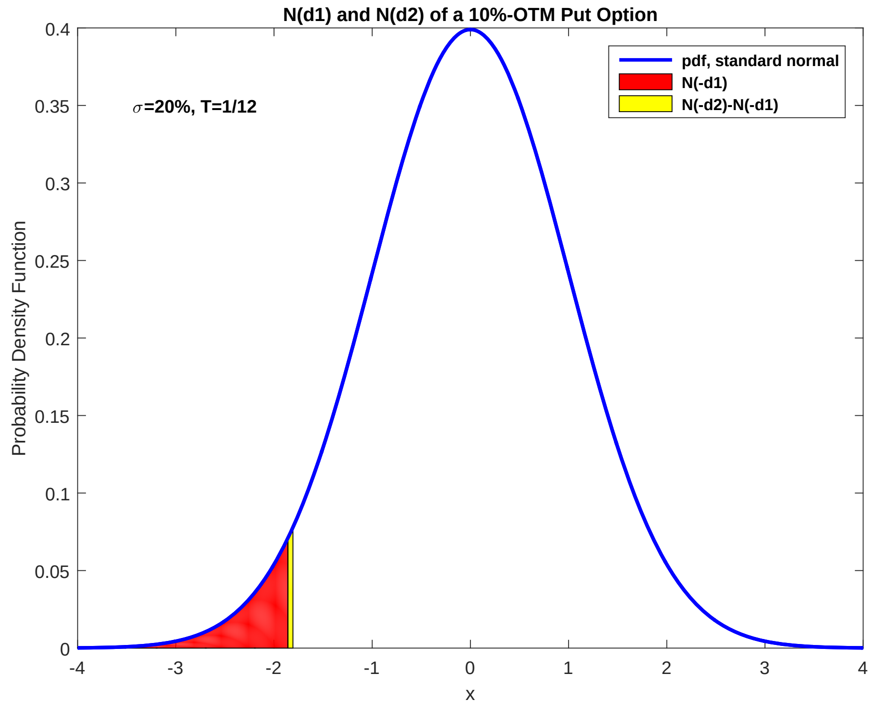

Figure 4: The Distribution of a Standard Normal with $N(-d_1)$ and $N(-d_2)$.

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above pricing into:

$$\frac{P_0}{S_0} = \frac{e^{-rT}K}{S_0}\boxed{N(-d_2)} - \boxed{N(-d_1)} = 0.90 \times \boxed{N(-d_2)} - \boxed{N(-d_1)}$$

Figure 4 gives us a graphical presentation of what matters when it comes to pricing such OTM options: the left tail in red and the slice in yellow. The areas in red and yellow are mapped directly to the CDF of a standard normal (hence $N(-d_1)$ and $N(-d_2)$ - $N(-d_1)$) because we are working under the Black-Scholes model. But the intuitive goes further. For any distribution (even it is not normal), what matters for the pricing of this OTM put option is the left-tail distribution. If this left tail is fat because of many financial crises, then the pricing of OTM put options should reflect these tail events. In Assignment 3, you will have a chance to work with a model with crash and see the link between fat tails and option prices.

As mentioned a few times, the actual distribution of stock market returns is not normally distributed. This is especially true for returns at higher frequencies (e.g., daily returns). As such, the Black-Scholes model fails to capture the fat tails in the data. As we will see, this becomes a rather important issue when it comes to pricing options.

Conversely, by looking at how these OTM options are priced, we learn about investors' assessment and attitude toward these tail events.

- **Option Implied Smirks:** After the 1987 stock market crash, one very robust pattern arose from the index options (SPX) market called volatility smiles or smirks.

  Consider the nearest term options, say one month to expiration (T=1/12). Let's vary the strike price of these options. Typically, for options with one month to expiration, you can find tradings of OTM puts and calls that are up to 10% out of the money. It is generally the case that OTM options are more actively traded than in-the-money options. This makes sense. If you are using options for speculations, you would prefer options that are cheaper (and are liquid) so that you can get more action for each dollar invested in options. If you are using options for hedging, it is likely that you are hedging out tails events. So either way, the OTM puts and calls are referred instruments than ITM options.

  Between OTM calls and puts of SPX, it is generally the case that OTM puts are more actively traded and the level of OTM-ness can reach up to 20%. For the S&P 500 index, a typical annual volatility is 20%, implying a monthly volatility of 5.77%. So for a 10% OTM put option, it takes a drop of 1.733-sigma (10%/5.77%) move in the S&P 500 index over a one-month period for this option to come back to the money.

  Using the market prices of all the available SPX puts and calls, we can back out the Black-Scholes implied volatility σ^I for each one of them. If investors are pricing the options according to the Black-Scholes model, then we should see σ^I being exactly the same for all of these options, regardless of the moneyness of the options. What we see in practice, however, is a pattern like that in Table 1.

  Table 1: Short-Dated SPX Puts with Varying Moneyness on March 2, 2006.

P_0	S_0	K	OTM-ness	T	σ^I	P_0^{BS}
9.30	1287	1285	0.15%	16/365	10.06%	?
6.00	1287	1275	0.93%	16/365	10.64%	5.44
2.20	1287	1250	2.87%	16/365	12.74%	0.92
1.20	1287	1225	4.82%	16/365	15.91%	0.075
1.00	1287	1215	5.59%	16/365	17.24%	0.022
0.40	1287	1170	9.09%	16/365	22.19%	0.000013

  Table 1 lists six short-dated OTM put options with exactly the same time to expiration but varying degrees of moneyness. The first option is nearest to the money, striking at $K = 1285$ when the underlying stock index is at $S_0 = 1287$. The last option is the farthest away from the money, striking at $K = 1170$. The S&P 500 index needs to drop by over 9% over the next 16 calendar days in order for this option to be in the money. Not surprisingly, options are cheaper as they are farther out of the money. But what's interesting is that their Black-Scholes implied vols exhibit this opposite pattern: the more out of the money a put option is, the higher its implied vol. In other words, even though the pricing of \$0.40 (per option on one underlying share of the S&P 500 index) seems very cheap in dollars and cents, it is actually over priced. Plugging a $\sigma = 10.06\%$ to the Black-Scholes model (which a closer to the market volatility around March 2, 2006), the model price for this OTM put is \$0.000013. In other words, this option is so out of the money, the Black-Scholes model (with normal distribution) deems its value to be close to zero. In practice, however, there are people who are willing to pay \$0.40 for it.

Why? Don't they know about the Black-Scholes option pricing formula? If they care about tail events, then what about OTM calls which are sensitive to right tails? As we see in the data, the tail fatness shows up in both the left and the right. But the OTM calls are not over-priced. If anything, the implied vols of OTM calls are on average slightly lower than ATM options. That is why we are calling this pattern volatility smirk, which is an asymmetric smile.

- **Expected Option Returns:** Another way to look at the profit/loss involved in options is to calculation their expected returns like we do in the stock market. Table 2 was reported in a 2000 *Journal of Finance* paper by Prof. Coval and Shumway.

Table 2: Expected Options Returns

Strike - Spot	-15 to -10	-10 to -5	-5 to 0	0 to 5	5 to 10
Weekly SPX Put Option Returns (in %)					
mean return	-14.56	-12.78	-9.50	-7.71	-6.16
max return	475.88	359.18	307.88	228.57	174.70
min return	-84.03	-84.72	-87.72	-88.90	-85.98
mean BS β	-36.85	-37.53	-35.23	-31.11	-26.53
corrected return	-10.31	-8.45	-5.44	-4.12	-3.10

Option data from Jan. 1990 through Oct. 1995.

As shown in Table 2, the weekly returns of buying put options are on average negative. There are quite a bit of variation in these returns. For the farther OTM put options, the return could be as positive as 475.88%, or as negative as -84.03%. This option has a beta of -36.85, which is due to the inherent leverage of these options. The CAPM-alpha of this investment is -10.31% per week. Whoever is selling this option would make a lot of money...on average. But he needs to be well capitalized when an event like 475.88% happens.

Calculations like those in Table 2 are rather imprecise because of the large variations in option returns. So we do not want to take the numbers too literally. But the qualitative result of this Table is important: when it comes to investing in options, there are large variations in option returns. Moreover, buying put options give you negative alpha. The more out of the money the put option is, the more negative the alpha becomes. For investors who are selling such put options, they are able to capture such alpha. But such trading strategies are in generally very dangerous. You need to be well capitalized to survive large crises like the 1987 stock market crash. Otherwise, you are just one crisis away from bankruptcy.

The results shown here in the return space is very much consistent with the earlier results in the implied-vol space, where OTM put options are over priced relative to near-the-money options. The level of over-pricing gets more severe as the put option becomes more out of the money and are more sensitive to market crashes. So it is not surprising that the put option returns are on average negative. Most of the times, you purchase an insurance against a market crash, but the crash does not happen and your put option expires out of the money. But once in a while, a crisis like 1987 or 2008 happens, then this put option brings you over-sized returns. Sitting on the other side of the trade are investors who sell/write you these crash insurances. Most of the times, they are able to pocket the premiums paid for the insurance without having to do anything. But once in while, they lose quite a bit of money if a crisis like 1987 or 2008 happens. As such, the risk profile of such option strategies differs quite significantly from that of a stock portfolio, where all instruments are linear. In Assignment 3, you will have a chance to see this kind of risk/return tradeoff of options in more details for yourselves.

2 When Crash Happens

- **Crash and Crash Premium:** The empirical evidence we've seen so far indicates that strategies involving selling volatility and selling crash insurance are profitable. As you will see for yourself in Assignment 3, the return distribution of such option strategies differs quite significantly from that of a stock portfolio, where all instruments are linear.

In the presence of tail risk, options are no longer redundant and cannot be dynamically replicated. As such, two considerations involving the tail risk become important in the pricing of options. First, the likelihood and magnitude of the tail risk. Second, investor's aversion or preferences toward such tail events. The "over-pricing" of put options on the aggregate stock market (e.g., the S&P 500 index) reflects not only the probability and severity of market crashes, but also investors' aversion to such crashes — crash premium.

In fact, as you will see in Assignment 3, the probability and severity of market crashes implicit in the volatility smirk are such that investors are pricing these OTM put options as if crashes like 1987 would happen at a much higher frequency. In other words, investors are willing to pay a higher price for such crash insurances even though they are "over-priced" relative to the actual amount of tail risk observed in the aggregate stock market. And the sellers of such crash insurances are only willing to sell them if they are being compensated with a premium, above and beyond the amount of tail risk in the data. This crash premium accounts for most of the "over-pricing" in short-dated OTM puts and ATM options.

By contrast, this "over-pricing" is not severe for OTM calls because they are not very sensitive to the left tail. Instead, OTM calls are sensitive to the right tail. From how such options are priced relative to OTM puts, it is obvious that investors are not eager to pay the same amount of premium for insurances against the right tail. This makes perfect sense. The intuition comes straight from the CAPM. An OTM call is a positive beta security, which provides positive returns when the market is doing well. It is icing on the cake. By contrast, an OTM put pays when the market is in trouble — a friend in need is a friend indeed.

- **Bank of Volatility:** LTCM was a hedge fund initially specialized in fixed-income arbitrage. It was extremely successful in its earlier years. Success breeds imitation. Soon, the fixed-income arbitrage space was crowded and spreads in arbitrage trades were shrinking. In early 1998, LTCM began to short large amounts of equity volatility. Betting that implied vol would eventually revert to its long-run mean of 15%, they shorted options at prices with an implied volatility of 19%. Their position is such that each percentage change in implied vol will make or lose $40 million in their option portfolio. The size of their vol position was so big that Morgan Stanley coined a nickname for the fund: the Central Bank of Volatility. For more details, you can read

Roger Lowenstein's book on LTCM.

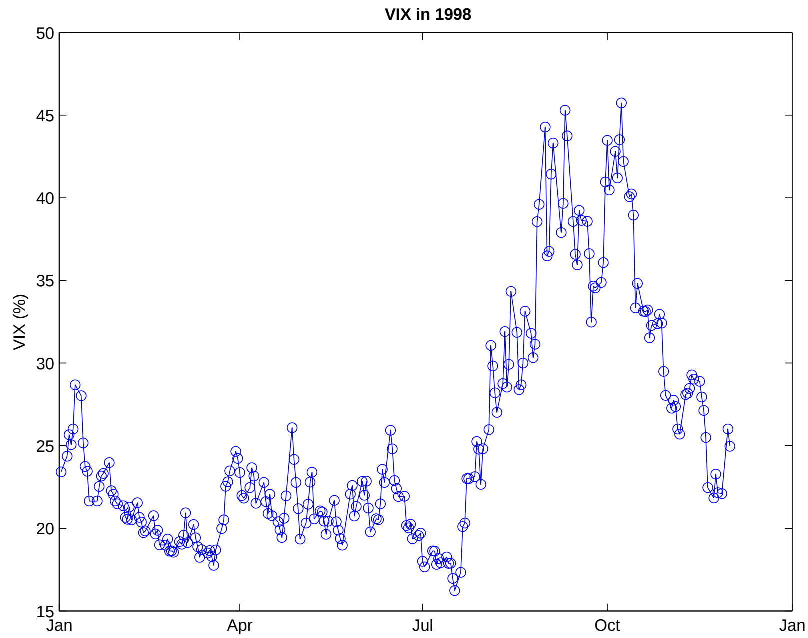

Figure 5: Time Series of CBOE VIX index in 1998.

During normal time, volatility does revert to its mean. So the idea behind the trade makes sense. Moreover, as we've seen in the data, selling volatility (via ATM options) is a profitable strategy on average because of the premium component. But the premium was not a free lunch: it exists become of the risk involved in selling volatility. As we've seen in the data, when the volatility of the aggregate market suddenly spikes up, the financial market usually is in trouble. Whenever the market is in the crisis mode, there is flight to quality: investor abandon all risky asset classes and move their capital to safe havens such as the Treasury bond market.

For the case of LTCM in 1998, it had arbitrage trades in different markets (e.g., equity, fixed-income, credit, currency, and derivatives) across different geographical locations (e.g., U.S., Japan, and European). Lowenstein's book gives more detailed descriptions of these arbitrage trades. One common characteristics of these arbitrage trades is that they locate some temporary dislocation in the market and speculate that this dislocation will die out as the market converges back to normal. In a way, these arbitrage trades betting on convergence make money because they provide liquidity to temporary market dislocations. The key risk involved in these arbitrage trades is that timing of the convergence is uncertain. Sometimes, instead of converging, the dislocation becomes even more severe before converging back to normal.

Prior to the Russian default in the summer of 1998, these arbitrage trades were not highly correlated. But after the default, most of these previously uncorrelated arbitrage trades lost money for LTCM at the same time. This certainly includes the volatility trades. As shown in Figure 5, early in the year, volatility was fluctuating around 20%. By summer 1998, however, the market became quite volatile because of the Russian default. At its peak, the VIX index was around 45%. Recall that LTCM was selling volatility when VIX was around 19% in early 1998. The position was such that each percentage change in implied vol will make or lose \$40 million. So if the volatility converges back to its long run mean of 15%, then roughly $4 \times \$40 = \$160$ can be made. But if instead of converging, the volatility increases to 45%, you can imagine the loss.

The Russian default affected not only LTCM but other hedge funds and prop trading desks who pursued the same kind of convergence trades. At a time like this, capital becomes scarce, and all leveraged investors (e.g., hedge funds or prop trading in investment banks) are desperately looking for extra source of funding. They do so by unwinding some of their arbitrage trades, further exacerbating the widening spreads. At a time like this, holding a security that pays (e.g., an existing long position in puts) could be very valuable. By contrast, a security that demands payment (e.g., an existing short position in puts) would be threatening to your survival. Therefore, being on the short side of the market volatility hurts during crises. That is why volatility is expensive (i.e., ATM options are over-priced) in the first place.

- **The 2008 crisis:** The OTM put options on the S&P 500 index is a good example for us to understand crash insurance. In writing a deep OTM put option, the investor prepares himself for the worse case scenario when the option becomes in the money. This happens when the overall market experiences a sharp decline. The probability of such events is small. But if he writes a lot of such options believing that the exposure can somehow be contained by the low probability, then he is up for a big surprise when a crisis does happen. As we learned from the recent financial crisis, some supposedly sophisticated investors wrote such OTM put options without knowing the real consequence.

  Gillan Tett from *Financial Times* wrote an excellent book called *Fool's Gold* with details of how investment banks developed and later competed for the market shares of the mortgage-linked CDO products. The following is a brief summary.

  By 2006, Merrill, who was late into the CDO game, topped the league table in terms of underwriting CDO's, selling a total of $52 billion that year, up from $2 billion in 2001. Behind the scenes, Merrill was facing the same problem that worried Winters at J.P. Morgan: what to do with the super-senior tranch?

CDO's are the collateralized debt obligations. It pools individual debt together and slices the pool into tranches according to seniority. For a mortgage-linked CDO, the underlying pool consists of mortgages of individual homeowners. The cashflow to the pool consists of their monthly mortgage payments. The most senior tranch is the first in line to receive this cashflow. Only after the senior tranch receives its promised cashflow, the next level of tranches (often called mezzanine tranches) can claim their promised cashflow. The equity tranch is the most junior and receives the residual cashflow from the pool.

As default increases in mortgages, the cashflow to the pool decreases. The equity investors will be the first to be hit by the default. If the default rate further increases, then the mezzanine tranch will be affected. The most senior tranch will only be affected in the unlikely event that both equity and mezzanine investors are wiped out and the cashflow to the pool cannot meet the promise to the most senior tranch. Such super senior tranches are usually very safe and are Aaa rated. By contrast, the mezzanine tranches are lower rated (Baa) because of the higher default risk. And the credit quality of the equity tranch is even lower.

The pricing of such products is consistent with their credit quality: the yield on the mezzanine tranches is higher than the senior tranches to compensate for the higher credit risk. Investors, in an effort to reach for yield, prefer to buy the mezzanine and equity tranches. As a result, the investment banks underwriting the CDOs are often stuck with the super senior tranches. As the business of CDOs grew, the banks are accumulating more and more highly rated super senior tranches. Initially, Merrill solved the problem by buying insurance (credit default swaps) for its super-senior debt from AIG.

Let's take a look at what the super-senior tranch is really about. It is highly rated because of the low credit risk. Imagine the economic condition under which this credit risk affecting the super-senior tranch will actually materialize: when the default risk is so high that both mezzanine and equity investors are wiped out. A typical argument for the economics of pooling is that default risk by individual homeowners can be diversified in a pool. This is indeed true when we think about the risk affecting the equity tranch: one or two defaults in the pool would affect the cashflow to the equity tranch, but would not affect the mezzanine tranch, let alone the senior tranch. So the risk affecting the senior tranch has to be a very severe one. The default rate has to be so high that the cashflow dwindles to the extent that it would eat through the lower tranches and affect the most senior tranch. In other words, many homeowners must be affected simultaneously and default at the same time to generate this type of scenario. By then, the risk is no longer idiosyncratic but systemic. So writing an insurance on a senior tranch amounts to insuring a crisis — a deep OTM put option on the entire economy.

In late 2005, AIG told Merrill that it would no longer offer the service of writing insurance on senior tranches. By then, however, AIG has already accumulated quite a large position on such insurance. Later, AIG was taken over by the US government in a $85 billion bailout and the insurance on senior tranches was honored and made whole by AIG (and the New York Fed).

After AIG declined to insure their super senior tranch, Merrill decided to start keeping the risk on its own books. At the same time, Citigroup, another late comer, was also keen to ramp up the output of its CDO machine. Unlike the brokerages, though, Citi could not park unlimited quantities of super-senior tranches on its balance sheet. Citi decided to circumvent that rule by placing large volumes of its super-senior in an extensive network of SIVs (Special Investment Vehicle) and other off balance sheet vehicles that it created. Citi further promised to buy bank the super-senior tranch if the SIVs ever ran into problems with them.

Now let's try to understand what Merrill and Citi are actually doing by retaining the super-senior tranch. Effectively, they are holding the super-senior tranch without an insurance. If you are holding a US treasury bond, you don't have to worry about credit risk (except for when the US government defaulted). So holding a super-senior tranch without an insurance is like holding a default-free US treasury bond and selling a deep OTM option on the overall economy at the same time. Before, they were able to buy that put option from AIG to hedge out this risk. Now, they are bearing this risk themselves.

Then the crisis happened in 2007 and 2008, and the mortgage default rate increased to such an extent that it started to affect the super-senior tranches. In other words, the deep OTM put options became in the money. During the 2007-08 crisis, the pricing of these super-senior tranches became one of the biggest headaches on Wall Street. Merrill and Citi, along with other Wall Street banks, had to take billions of dollars of writedowns.

3 Beyond the Black-Scholes Model

- **A model with market crash:** In Assignment 3, you will be working closely with a model that allows market to crash. It is a simplifies version of the model in Merton (1976).
- **A model with stochastic volatility:** I'll briefly mention these models in class.

Appendix

During my office hours, I got a few questions about the Brownian motion and risk-neutral pricing. Let me use this appendix to explain some of the details.

A Brownian Motion

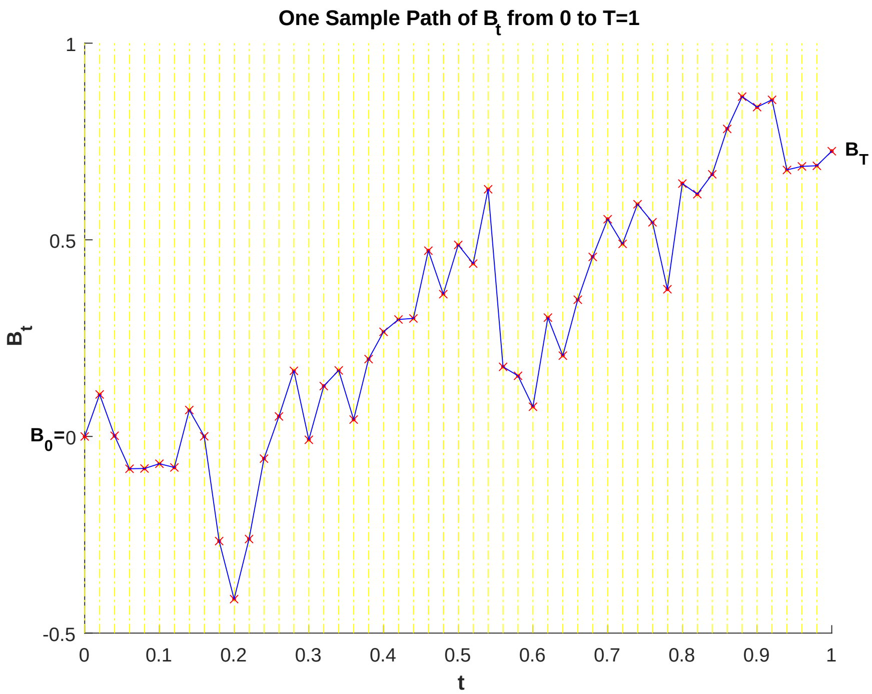

Figure 6: One sample path of a Brownian motion.

To understand the Brownian motion, let's create one. Let's start from time 0 and end in time T. Let's further chop this time interval into small increments. For example, in Figure 6, T=1 and the interval between 0 and 1 is chopped evenly into 50 smaller increments with size $\Delta = 1/50$. We can now start to create a sample path of the Brownian motion:

$$
\begin{aligned}
&B_0 = 0 \\
&B_\Delta - B_0 = \sqrt{\Delta}\,\epsilon_\Delta \\
&B_{2\Delta} - B_\Delta = \sqrt{\Delta}\,\epsilon_{2\Delta} \\
&\dots \\
&B_T - B_{T-\Delta} = \sqrt{\Delta}\,\epsilon_T\,.
\end{aligned}
$$

where the ϵ's are independent standard normals. In creating this sample path, we use the first two properties of the Brownian motions: independence increments and stationary normal increments. I've attached the Matlab code I used to create this plot in this note. You can run it and each time you will get a different sample path.

For our purpose of pricing a European-style option, what matters is the distribution of B_T. But if we are interested in pricing an American-style option, then the entire path of B_t matters and at each node, we will make a decision of whether or not to exercise early. So the grid should be as fine as possible (larger N and smaller Δ).

Now back to our original process for S_t:

$$dS_t = \mu\, S_t\, dt + \sigma\, S_t\, dB_t\,,$$

where to avoid distraction, I have set the dividend yield $q = 0$. As usual, we work with $X_t = \ln S_t$ and, using the Ito's Lemma, we have

$$dX_t = \left(\mu - \frac{1}{2}\sigma^2\right)\, dt + \sigma\, dB_t\,.$$

The nice thing about working with $\ln S_t$ is that you can integrate out the process:

$$\begin{aligned} X_T &= X_0 + \int_0^T \left(\mu - \frac{1}{2}\sigma^2\right)\, dt + \int_0^T \sigma\, dB_t \\ &= X_0 + \left(\mu - \frac{1}{2}\sigma^2\right)\, T + \sigma\, \left(B_T - B_0\right) \\ &= X_0 + \left(\mu - \frac{1}{2}\sigma^2\right)\, T + \sigma\, \sqrt{T}\, \epsilon_T\,, \end{aligned}$$

where in the last step I use $\sqrt{T}\,\epsilon_T$ to express $B_T - B_0$. Recall that the log-return R_T is defined by $R_T = \ln X_T - \ln X_0$. We have

$$R_T = \left(\mu - \frac{1}{2}\sigma^2\right)\, T + \sigma\, \sqrt{T}\, \epsilon_T$$

B Change of Measure, Risk-Neutral Pricing

Under the original measure (P-measure), the process runs as

$$dX_t = \left(\mu - \frac{1}{2}\sigma^2\right)\, dt + \sigma\, dB_t\,.$$

If we were to do option pricing under this measure, we know that we cannot do

$$C_0 \neq e^{-rT} E\left(S_T - K\right)^+ \ .$$

This is a big no-no in Finance because it approaches the pricing as if we were risk neutral. Interestingly, this is why the method of "risk-neutral" pricing arises. It is mostly a mathematical result. If you Google Girsonov theorem or the Radon-Nikodym derivative, you will see the related math result. But the math result has its relevance in Finance. Let me approach it this way.

In Finance, we develop this concept of pricing kernel or the stochastic discount factor. Armed with this pricing kernel ξ_T, we can do our pricing:

$$C_0 = e^{-rT} E\left(\frac{\xi_T}{\xi_0}\left(S_T - K\right)^+\right) \ .$$

Under the Black-Scholes setting, the markets are complete and the pricing kernel is unique. In fact, as an application of the Girsonov theorem, this pricing kernel is of the form

$$\xi_T = \frac{dQ}{dP} = e^{-\gamma\, B_T - \frac{1}{2}\gamma^2\, T} \ .$$

This ξ_T is what the mathematician would call the Radon-Nikodym derivative. Notice that by construction $E(\xi_T) = 1$.

As mentioned earlier, the pricing kernel is unique under the Black-Scholes setting. So the constant γ is uniquely defined. In Finance, we call this parameter the market price of risk and for the Black-Scholes setting, it is $\gamma = \left(\mu - r\right)/\sigma$, which in fact is the Sharpe ratio. In a more general setting, γ can itself be a stochastic process. Also notice that with a positive market price of risk, $\gamma > 0$, ξ_T is negatively correlated with B_T (hence negatively correlated with X_T and S_T). This is what you were taught in Finance 15.415. When S_T experiences a positive stock, the stochastic discount factor is smaller; when S_T experiences a negative stock, the stochastic discount factor is bigger. This asymmetry has its origin in the fact that investors are risk averse and the risk in S_T is systematic (undiversifiable).

It turns out that we can create a new measure Q, called the equivalent martingale measure, for the original P and the pricing becomes,

$$\begin{aligned} C_0 &= e^{-rT} E\left(\frac{\xi_T}{\xi_0}\left(S_T - K\right)^+\right) \\ &= e^{-rT} E^Q\left(\left(S_T - K\right)^+\right) \ , \end{aligned}$$

and the link between these two measures is $\xi_T = dQ/dP$.

Now let's construct this new Q-Brownian:

$$\begin{aligned} dX_t &= \left(\mu - \frac{1}{2}\,\sigma^2\right)\, dt + \sigma\, dB_t^p \\ &= \left(r - \frac{1}{2}\,\sigma^2\right)\, dt + \sigma\left(\frac{\mu - r}{\sigma} + dB_t^P\right) \\ &= \left(r - \frac{1}{2}\,\sigma^2\right)\, dt + \sigma\, dB_t^Q\,, \end{aligned}$$

where the Q-Brownian is defined as

$$dB_t^Q = \frac{\mu - r}{\sigma} + dB_t^P\,.$$

And this change of measure, from P to Q, is the essence of the risk-neutral pricing.

The name of "risk-neutral" pricing is ironical: the whole thing arises from the observation that we cannot do

$$C_0 \neq e^{-rT} E^P\left(S_T - K\right)^+\,.$$

But if we are willing to change our probability measure from P to Q, under which

$$dX_t = \left(r - \frac{1}{2}\,\sigma^2\right)\, dt + \sigma dB_t^Q\,,$$

then we can indeed do

$$C_0 = e^{-rT} E^Q\left(S_T - K\right)^+\,.$$

C Change of Measure, One More Application

This mathematical tool can be further exploited. Recall that we need to do this calculation in our Black-Scholes option pricing,

$$e^{-rT}\, E^Q\left(S_T \mathbf{1}_{S_T > K}\right)$$

What if we can drop S_T and change it to

$$S_0\, E^{?}\left(\mathbf{1}_{S_T > K}\right)$$

That would make our math very simple.

In fact, we can drop S_T like the way we dropped ξ_T. As long as the process is positive, there is an equivalent martingle measure waiting for us to help us simplify the math. This is where the new measure QQ comes from. You can start with the observation that

$$S_T = e^{X_T} = e^{\sigma\, B_T + \text{other deterministic terms}}$$

You can then check

$$\begin{aligned} dX_t &= \left(r - \frac{1}{2}\,\sigma^2\right)\, dt + \sigma dB_t^Q \\ &= \left(r + \frac{1}{2}\,\sigma^2\right)\, dt - \sigma^2\, dt + \sigma dB_t^Q \\ &= \left(r + \frac{1}{2}\,\sigma^2\right)\, dt + \sigma\, \left(-\sigma\, dt + dB_t^Q\right) \end{aligned}$$

So if we define

$$dB_t^{QQ} = -\sigma\, dt + dB_t^Q\,,$$

under which

$$dX_t = \left(r + \frac{1}{2}\,\sigma^2\right)\, dt + \sigma dB_t^{QQ}\,.$$

Then we can indeed get

$$e^{-rT}\, E^Q\left(S_T \mathbf{1}_{S_T > K}\right) = S_0 E^{QQ}\left(\mathbf{1}_{S_T > K}\right)\,.$$

I am being a bit sloppy in my notation, but I trust a careful and thorough student would fill in the details (including the *other deterministic terms*).

D Matlab Code

Code 1: Brownian.m

```
T=1; N=50;

Delta=T/N;
EPS=randn(N,1);
T_vec=(0:Delta:T)';

B=0; B_vec=B;
```

```
for i=1:N,
  B=B+EPS(i)*sqrt(Delta);
  B_vec=[B_vec; B];
end

figure(1); clf; hold on;
plot(T_vec,B_vec,'r.');
BND=axis;
for i=1:N,
 plot(T_vec(i)*[1 1],BND(3:4),'y-.');
end
plot(T_vec,B_vec,'rx',T_vec,B_vec,'b-');
hold off;
ylabel('\bf B_t');
xlabel('\bf t');
text(1.01,B,'\bf B_T');
text(-0.08,B_vec(1),'\bf B_0=');
title('One Sample Path of B_t from 0 to T=1')
```