

Class 21: Fixed Income, Yield Curve

This Version: November 17, 2017

1 Factors Influencing the Yield Curve

- **The Yield Curve:** The Treasury yield curve is the best way to summarize the market prices of Treasury bonds, just like the implied-vol curves in the options market. In options, the vol curves become a three-dimensional surface because of an option could vary in its moneyness as well as time to expiration. In bonds, the yield curve remains a two-dimensional curve: a plot of yield against maturity.

Of course, bonds of the same maturity also vary in their “moneyness”: new bonds are issued at par with $y = c$ and $P = \$100$; old bonds issued during high interest-rate environment are premium bonds with $c > y$ and $P > \$100$; bonds issued during extremely low interest-rate environment will eventually become discount bonds with $c < y$ and $P < \$100$. Because of this, when we talk about yield curve, we need to be more specific. In general, for coupon bonds, we usually use the par curve: the yields for par coupon bonds. For a given maturity, the yield of a par coupon bond will be located ... exactly on the curve, while the yields for discount/premium bonds will be close to the curve but slightly off.

With an upward sloping yield curve, the yield of a premium bond sits below the par curve while the yield of a discount bond sits above the par curve. This is, because the premium bond, with relatively higher coupon payments $c^{\text{premium}} > c^{\text{par}} > c^{\text{discount}}$, puts a relatively higher weight on the yields of shorter maturities. With an upward sloping yield curve, this translates to a slightly lower yield. Overall, however, the differences are not huge. I would encourage you to go through the math yourself to verify this intuition and gauge the magnitude.

In doing these calculations, there is always a curve in the background that guides our intuition. That is the zero curve, which is effectively the collection of discount functions over different maturities. Having this zero curve is useful in discounting cashflows and

we can price bonds of all maturities and varying coupon rates, using the discount function dictated by the zero curve. As a result, this zero curve enforces the pricing of bonds of all maturities to be internally consistency. Now the question is where do we get a zero curve in the first place? Most of the bonds in the market are coupon-bearing bonds, we do not observe the zero curve directly. So the common practice is to build a zero curve from market prices of coupon-bearing bonds. In fact, this task of yield curve fitting should be a very basic skill for a fixed-income person. If I have time for the next class (on term structure modeling), I'll talk more about the exact approach. By the way, using the intuitive developed earlier about premium/discount bonds, we know that, with an upward sloping yield curve, the zero curve sits above the par curve.

- **Factors Influencing the Yield Curve:** Our discussion so far focuses on the internal consistency of bond pricing. We imagine that there is a zero curve and ask the pricing of all coupon-bearing bonds to be consistent with this zero curve. The first question you would ask is: what are the factors influencing this zero curve? In an environment of constant interest rate, this zero-curve would always be flat. Then there is not too much to talk about. In practice, the curve is not flat and interest rates are not constant. So what can we learn about them? We will try to answer this question in today's class.

Once we are happy with the answers to the first question, we will ask the second question, which is also very interesting. With a zero curve (or even a sophisticated term structure model), we price all coupon-bearing bonds traded in the market. How good is our curve (or model) in pricing all of these bonds? What are we to learn when some bonds are mis-priced by a curve (or model)? We will try to answer this question in our next class.

As you can see, both questions focus on the same issue: what are the factors influencing bond pricing in the market place? I split the question into two so that we can answer this important question in two steps. First, we address the economic factors influencing the yield curve. In a way, these factors are more macro and systematic, affecting every "body" on the yield curve. Second, we address the institutional reasons affecting the yield curve. These factors are more localized and idiosyncratic.

- **Movements and Co-Movements in Yield Curve:** Figure 1 plots the time-series of Treasury yields of the representative maturities: 3M, 2Y, 5Y, 10Y, and 30Y. As you can see, most of the time, the yield curve is upward sloping. In a few occasion, the yield curve becomes flat or even inverts to a downward sloping curve.

It is also evident from the plot that there is quite a bit of comovement in yields across

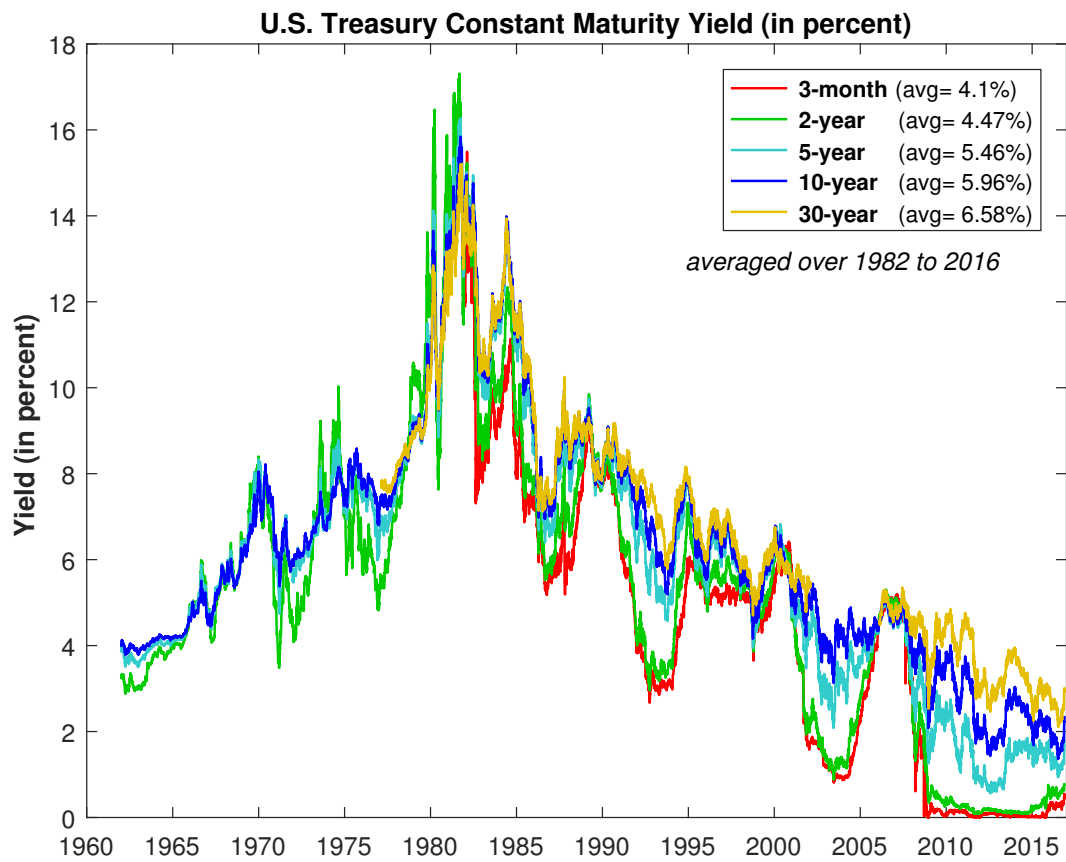


Figure 1: Time-Series of Treasury Constant Maturity Yields.

Table 1: Comovement in Yields, Daily Data from 1982 to 2015

corr in yields (%)					
	3M	2Y	5Y	10Y	30Y
3M	100.0	98.57	96.19	93.61	90.90
2Y	98.57	100.0	99.18	97.54	95.47
5Y	96.19	99.18	100.0	99.46	98.19
10Y	93.61	97.54	99.46	100.0	99.57
30Y	90.90	95.47	98.19	99.57	100.0
corr in daily changes in yields (%)					
	3M	2Y	5Y	10Y	30Y
3M	100.0	57.31	46.87	40.18	35.15
2Y	57.31	100.0	90.29	82.17	72.90
5Y	46.87	90.29	100.0	94.07	85.74
10Y	40.18	82.17	94.07	100.0	93.71
30Y	35.15	72.90	85.74	93.71	100.0

different maturities. This is better summarized in Table 1. The pairwise correlations between yields of different maturities are well above 90%. Given the visible time trend and the persistence in yield (the auto-correlation in yield is close to 1), it is more meaningful to measure the comovement in changes in yields. After all, it is the surprise components (i.e., the random shocks) in yield that interest us the most. Measuring the pairwise correlations between daily changes in yields, we still find substantial comovements. Within the Treasury bonds and notes, the correlations between the two nearest maturities are above 90%. The comovement becomes relatively weaker as the maturities are further apart. But even for the 2Y and 30Y bonds, the correlation is around 70%. The connection between the Treasury bills and the rest of the yield curve is relatively weaker but still substantial: the correlation between 3M TBill and 2Y bond is about 57%.

Overall, we can see that the Treasury yield curve is an inter-connected curve. It is not a curve with its individual components moving around freely without any regard for other parts of the curve. In this sense, the curve is a tight family of individual members. But it is also not a curve with its individual components moving in exactly the same pace. There is some internal consistency and relationship. The closer the maturity, the stronger the relationship. Let's try to figure out the economic factors that drive these movements and comovements.

- **Monetary Policy and Fed Funds Rate:** By far, the most important factor in-

fluencing the yield curve is monetary policy. In the US, monetary policy is carried out by the Federal Reserve through the Federal Open Market Committee (FOMC). In 1977, Congress set explicit objectives for monetary policy: “maximum employment” and “price stability.” These two objectives in the Fed’s so-called dual mandate are not always in alignment and the committee members of FOMC face the task of making the right decision when these two objectives are in conflict with each other.

The fed funds rate is the main policy tool of the Fed. It is the rate at which depository institutions lend excess reserve balances to each other overnight. Bank reserves are funds that banks hold at the Fed, much like the checking accounts that individuals have at banks. A bank can use its reserve account at the Fed for making or receiving payments from other banks, as well as a place to hold extra cash. Banks are legally required to hold a minimum level of reserves. If a bank finds itself with reserve balances in excess of the required minimum level, it often lends the excess reserves out to other banks in the so-called fed funds market, in the form of an unsecured overnight loan. And the interest rate of this private loan is the fed funds rate.

Quoting the former Chairman Ben Bernanke, “Although the federal funds rate is a private rate between banks, the Fed was able to control it indirectly by affecting the supply of funds available to banks. More precisely, the Fed managed the funds rate by affecting the quantity of bank reserves.”

“The Fed was able to affect the quantity of bank reserves in the system, and thereby the federal funds rate, by buying or selling securities. When the Fed sells securities, for example, it gets paid by deducting their price from the reserve account of the purchaser’s bank. The Fed’s securities sales consequently drain reserves from the banking system. With fewer reserves available, banks are more eager to borrow from other banks, which puts upward pressure on the federal funds rate, the interest rate that banks pay on those borrowings. Similarly, to push down the federal funds rate, the Fed would buy securities, thereby adding to reserves in the banking system and reducing the need of banks to borrow from each other.”

Effectively, the Fed’s balance sheet is like a gigantic balloon attached to the entire US banking system. If the Fed feels that the economy is at the risk of overheating (i.e., the risk of high inflation), it will suck some air out of the system by selling securities into the system and therefore draining cash out of the banking system. If the Fed feels that the economy is performing poorly (i.e., the risk of high unemployment), it will blow some air into the system by buying securities from the system and therefore replenish the banking system with more cash.

In each of the FOMC meetings, the committee members weight the option of tightening (rate hike), loosening (rate cut), or no action. Over the history of FOMC meetings, the committee members are not always in agreement in terms of the right policy action. Hence the term *hawk*, who puts a higher weight on keeping the inflation low and often biases toward a tighter monetary policy; and *dove*, whose concern with respect to inflation is not as strong and often biases toward a loosening monetary policy. Of course, each policy decision is an “organic” process, with committee members taking into account of the information available to them at the time. If you read the memoirs of the former chairmen (e.g., Greenspan and Bernanke), you will notice that each decision weights heavily in their memory and on their conscience. Such men and women perform a great service to society.

- **Monetary Policy, Historical Experiences:** Figure 2 plots the time-series of Treasury yield curve (2-year and 10-year) along with the historical information that is relevant for our understanding of the monetary policy: inflation rate, GDP growth, and fed fund rates (the black solid line starting in the 1990s).

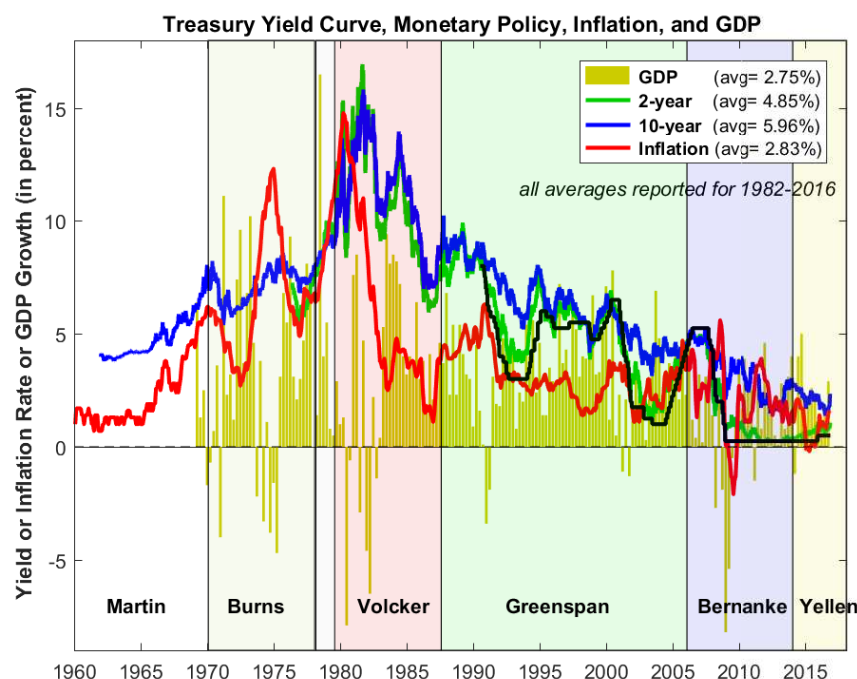


Figure 2: Treasury Yield Curve, Monetary Policy, Inflation Rate and GDP Growth.

Instead of writing about the historical experiences accompanying the events in Figure 2,

let me use the following extended excerpt from Ben Bernanke's book. I find it to be very useful in my own understanding of the US monetary policy, especially for the pre-Greenspan era of Volcker and Burns. It was a period over which I know very little about because ... I was still in China and thought of interest rate as little rectangular stamps collected in a little booklet. Even that remotely related activity was performed only once or twice when my dad dragged a reluctant me, less than ten years of age, to a bank in an effort to educate me on the virtue of being frugal and the benefit of saving.

Throughout most of the 1990s the Fed presided over an economy with employment growing strongly and inflation slowly declining to low levels. The Fed was thus meeting both parts of its congressional dual mandate to pursue maximum employment and price stability. In contrast, when I arrived at the Fed (August 2002), we saw risks to both sides of our mandate. On the employment side, we had the jobless recovery to contend with. On the price stability side, we faced a problem unseen in the United States since the Depression – the possibility that inflation would fall too low or even tip into deflation, a broad decline in wages and prices.

In the past, the end of a recession had typically been followed by an improving jobs market. But during the two years after the recession that ended in November 2001, the U.S. economy actually lost 700,000 jobs, and unemployment edged up from 5.5 percent to 5.8 percent even as output grew. Many economists and pundits asked whether globalization and automation had somehow permanently damaged the U.S. economy's ability to create jobs. At the same time, inflation had been low and, with the economy sputtering, Fed economists warned that it could fall to 1/2 percent or below in 2003. Actual deflation could not be ruled out.

Worrying about possible deflation was a new experience for FOMC participants. Ever since the end of the Depression, the main risk to price stability had always been excessive inflation. Inflation spiraled up during the 1970s. Paul Volckers Fed ended it, but at a steep cost. Within a few months of Volckers becoming chairman in 1979, the Fed dramatically tightened monetary policy, and interest rates soared. By late 1981, the federal funds rate hit 20 percent and the interest rate on thirty-year fixed-rate mortgages topped 18 percent. As a consequence, housing, autos, and other credit-dependent industries screeched to a halt. A brief recession in 1980 was followed by a

deep downturn in 1981-82. Unemployment crested above 10 percent, a rate last seen in the late 1930s.

After succeeding Volcker in 1987, Alan Greenspan continued the fight against inflation, although he was able to do so much more gradually and with fewer nasty side effects. By the late 1990s, the battle against high inflation appeared to be over. Inflation had fallen to about 2 percent per year, which seemed consistent with Greenspan's informal definition of price stability: an inflation rate low enough that households and businesses did not take it into account when making economic decisions.

The Great Inflation of the 1970s had left a powerful impression on the minds of monetary policymakers. Michael Moskow, the president of the Federal Reserve Bank of Chicago when I joined the FOMC (August 2002), had served as an economist on the body that administered the infamous – and abjectly unsuccessful – Nixon wage-price controls, which had attempted to outlaw price increases. (Predictably, many suppliers managed to evade the controls, and, where they couldn't, some goods simply became unavailable when suppliers couldn't earn a profit selling at the mandated prices.) Don Kohn had been a Board staff economist in the 1970s under Fed chairman Arthur Burns, on whose watch inflation had surged. Greenspan himself had served as the chairman of President Ford's Council of Economic Advisers and no doubt shuddered to remember the Ford administration's ineffectual Whip Inflation Now campaign, which encouraged people to wear buttons signifying their commitment to taming the rising cost of living. With Fed policymakers conditioned to worry about too-high inflation, it was disorienting to consider that inflation might be too low. But it was a possibility that we would soon have to take seriously.

- **Fed Funds Rate and Yield Curve:** Beginning in 1994, the FOMC began announcing changes in its policy stance, and in 1995 it began to explicitly state its target level for the fed funds rate. As you can see in Figure 3, this aspect of monetary policy has an immediate impact on the Treasury yield curve, especially on the short end. In many instances, the bond market, in anticipation of the impending rate change, would price the event in advance. For example, on September 13, 2001, when the bond market re-opened on a limited basis after 9/11, the 3M Tbill rate dropped 52 bps from 3.26% to 2.74%, the one-year rate dropped 50 bps from 3.31% to 2.81%, and the two-year rate dropped 54 bps from 3.52% to 2.99%. It was not until four days later, on September

17, the Fed cut the fed fund target rate from 3.50% to 3%.

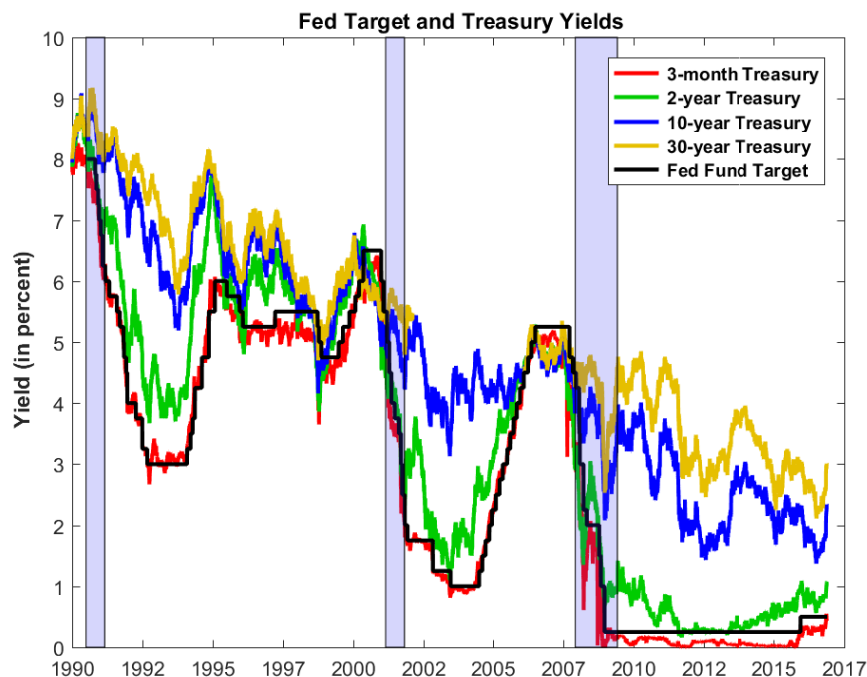


Figure 3: Treasury Yield Curve and Fed Fund Target Rate.

There is also a visible impact on the long-end of the yield curve, although the reactions of the longer end of the yield curve are not one-for-one in magnitude. This of course, makes sense given the impermanent nature of a monetary tightening or loosening. For example, On September 13, 2001, the five-year yield decreased by 38 bps from 4.41% to 4.03% and the ten-year yield decreased by 20 bps from 4.84% to 4.64%. Interestingly, the 30-year yield dropped by only 4 bps from 5.43% to 5.39%. (More on this topic on the 30-year yield next class.)

There were also times when the longer-term interest rates failed to rise after the Fed tightened monetary policy. This happened in 2004-05, the famous Greenspan's "conundrum." Quoting Bernanke again, "In speeches, I tied the conundrum to what I called the 'global savings glut' – more savings were available globally than there were good investments for those savings, and much of the excess foreign savings were flowing to the United States. Additional capital inflows resulted from efforts by (mostly) emerging-market countries like China to promote exports and reduce imports by keeping their currencies undervalued. To keep the value of its currency artificially low

relative to the dollar, a country must stand ready to buy dollar-denominated assets, and China had purchased hundreds of billions of dollars' worth of U.S. debt, including mortgage-backed securities."

Before leaving Figure 3, let me point out one well-known pattern: Prior to each of the three NBER-dated recessions, the yield curve was either very flat or even inverted. It turns out that the slope of the interest rate is a good predictor for future GDP growth.

- **Factors Influencing Monetary Policy:** Also plotted in Figure 3 are the NBER-dated recession periods. The cyclical nature of the monetary policy is obvious in this plot: tightening during economic expansions and loosening during recessions. It is also worthwhile to note that, in order to cause minimal disruption to the markets, the monetary policy applies itself to the market gradually. A typical rate cut/hike is in increments of 25 bps. There were four rate hikes that were 50 bps (twice in 1994, once in 1995 and 2000) and one rate hike of 75 bps (November 1994). There were sixteen rate cuts of 50 bps (three times in 1991-92, nine times in 2001-02 and four times in 2007-08) and three rate cuts of 75 bps (all happened in 2008).

Given the dual mandate of price stability and maximum employment, it is not surprising that expectations of the rate of inflation, GDP growth, and employment numbers (e.g., nonfarm payroll employment) influence the decision of the policy decision of the FOMC. The Stanford economist John B. Taylor wrote a paper in 1993, linking the policy rate explicitly to inflation rate and GDP. This became the famous Taylor rule and there are various extensions of this rule. Again, if you read the memoirs of Greenspan and Bernanke, you would see that each policy decision is an "organic" process, with committee members taking into account of the information available to them at the time. Having a mechanic rule is useful as a baseline, but cannot be the ultimate answer.

If you pay attention to the famous Wall Street activity called the "Fed Watch," you will notice market participants use all kinds of signal trying to predict the next policy move. Some macro investors also perform directional trades to express their views and they typically do so using the two-year notes. As such, the two-year yield are considered to be highly sensitive to changes in the Fed's policy outlook. Consequently, the shape of the yield curve (relative to the two-year yield) might contain information about the impending policy move. As you can see in Figure 3, the two-year yield has been increasing quite steadily since the beginning of 2015 in anticipation of the monetary tightening in the end of December 2015.

In addition, investors also use fed funds futures traded on CME to express their views.

Consequently, the pricing information in this market has been used to extract expectations of future Fed actions. This is a number watched closely by fixed-income traders and global macro investors. Even the Fed tracks this number to gauge the market expectation of their action. According to this calculation the implied probability of a rate hike from the current 25-50 bps to 50-75 bps is about 93.5%.

The market participants are involved in “Fed Watch” because uncertainties in the target rate have a big impact on the markets, not only the bond market but also the stock market (and the currency market). The Fed under chairman Bernanke and chairwoman Yellen has been working hard on Fed transparency in order to better communicate with the market participants in terms of the Fed policy.

- **Quantitative Easing and Operation Twist:** Earlier, we talked about how the Fed can use this gigantic balloon to suck/blow air into the entire banking system by selling/buying securities. Up to the 2008 crisis, the Fed performed monetary policy through affecting the fed funds rate. Starting from late 2008, the Fed employed a policy tool that is highly unorthodox and controversial: purchasing hundreds of billions of dollars of securities directly from the market with the intention to keep the long-term interest rates low.

After the FOMC meeting on October 29, 2008, the fed funds target rate was at 1% and the 3M Tbill rate was at 62 bps (the Treasury bill rates are lower than the overnight fed funds rate because of the potential counterparty risk involved in the unsecured fed funds loans). When the short-term interest rate reaches close to zero, what to do to bring down the longer-term interest rates in an effort to keep the economic recovery going? One way is to try to convince the market participants that the short-term interest rate will be kept low for a long time. In addition, the Fed also started to purchase securities in an effort to directly influence the long-term interest rate. On November 25, 2008, the Fed announced plans to perform large scale asset purchases, often referred to as “Quantitative Easing” or QE. As shown in Figure 4, the actual purchases happened in December 2008 for agency bonds (Fannie and Freddie debt) and January 2009 for mortgage-backed securities backed by Fannie Mae, Freddie Mac, and Ginnie Mae. This was later known as QE1, because of it was followed by QE2 and QE3.

From Figure 4, you can see that the Fed also purchased around \$300 billion in Treasury securities during QE1, partly to supplement the reduction in MBS holdings when the mortgages underlying the MBS were paid off (either because of home sales or

refinancings due the decreasing interest rate). We will visit this issue of negative convexity of MBS in a later class.

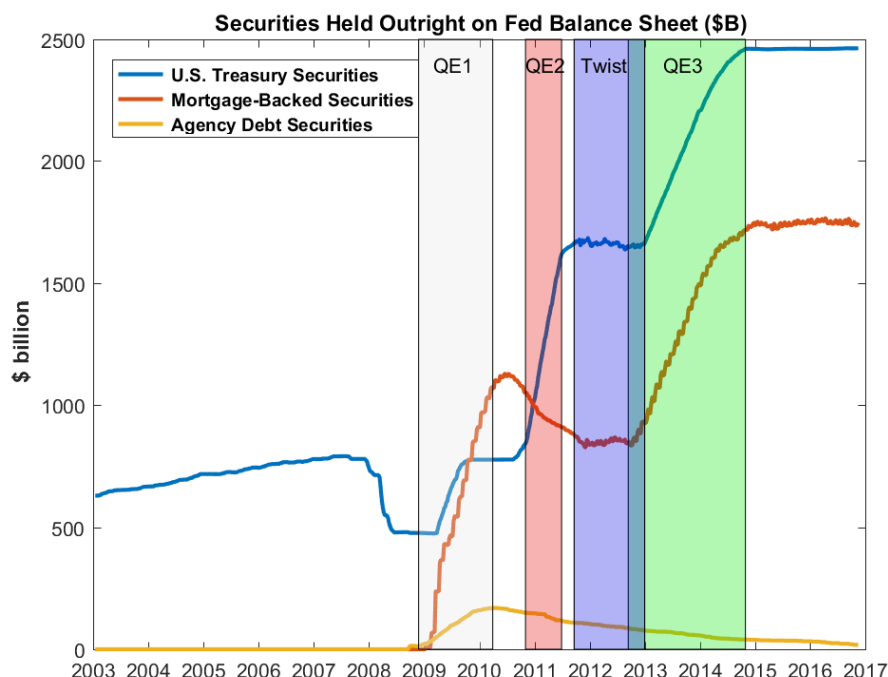


Figure 4: The Fed’s Balance Sheet.

QE1 was followed by QE2 and QE3 and a program called “operation twist” in between. The securities purchased through QE2 and QE3 can be seen through the Fed’s balance sheet in Figure 4. By now (November 11, 2015), the total market value of securities held outright on Fed’s balance sheet is \$4.24 trillion, with \$2.46 trillion in US Treasury securities. To put these numbers in perspective, let’s take a look at some other numbers. According to this Treasury website, as of August 2015, foreign holdings of the Treasury securities totals to \$6.099 trillion with China holding \$1.27 trillion and Japan holding \$1.197 trillion. According to the World Bank, the 2014 GDP is \$17.419 trillion for the US, \$10.360 trillion for China, \$4.601 trillion for Japan, \$3.852 trillion for Germany, and \$2.942 trillion for the UK.

Figure 5 plots the Fed’s holdings of Treasury securities by maturity. As shown Figure 5, during “Operation Twist,” the overall Treasury holding by the Fed remains nearly constant in market value. But the maturity of Fed’s holdings went through a big change. As shown in Figure 5, the Fed was actively selling Treasuries securities

maturing in 1-5 years and buying longer maturity bonds (5-10 years and longer than 10 years). Effectively, the Fed was increasing the duration of its Treasury portfolio without having to expand its balance sheet, in an effort to influence the long maturity yields so as to reduce the cost of credit for mortgage loans and corporate bonds.

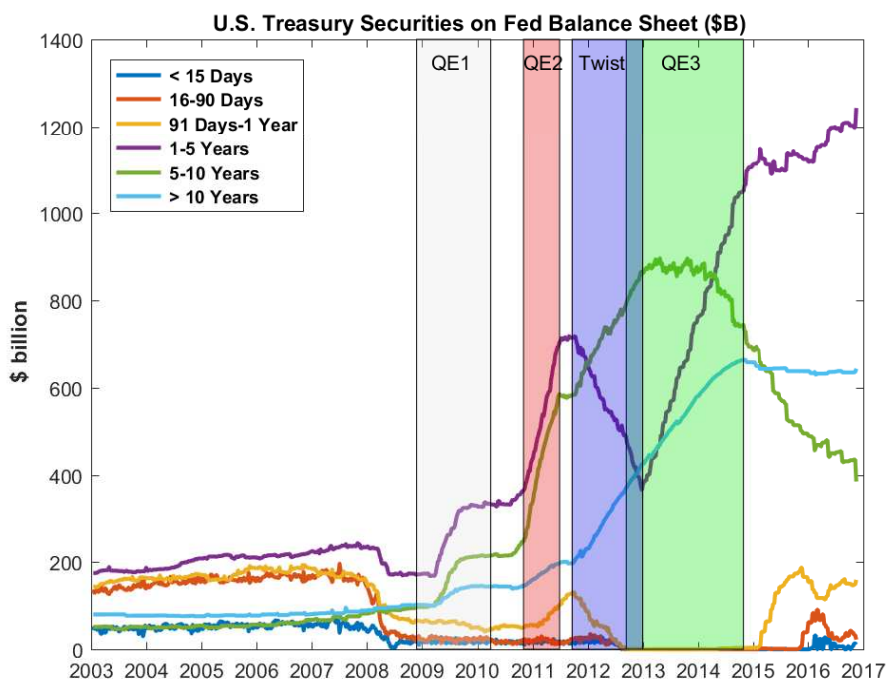


Figure 5: US Treasury Securities on Fed Balance Sheet, Maturity Composition.

The unconventional QE programs and the burgeoning Fed balance sheet were certainly not without risk. To put the policy thinking in perspective, let's take a look at the macro variables from 2008 to 2015 (see Figure 6). Prior to QE2, around October 2010, the unemployment rate was at 9.6%. The last time the unemployment rate was this high was during 1982-83 after the monetary tightening by Chairman Volcker. By contrast, the inflation was low at 1.1% in October 2010. Prior QE3, around August 2012, the employment rate was at 8.1% and the inflation was at 2%. It was clear that at the time, the Fed felt that the unemployment rates were too high (and inflation was not an issue of big concerns) and the economy needed help ... from somewhere. And the Fed's decision at the time was to step up and provide that help.

The economy has certainly been doing relatively better since then. As shown in Figure 6, by the end of QE3, the unemployment rate has been decreasing steadily to 5.7%

and the GDP growth was at 4.3%. Right now (October 2015), the unemployment rate is at 5% and the GDP growth has been uneven: 1.5% for the third quarter and 3.9% for the second quarter. Overall, however, it is difficult to quantify the effect of the QE programs. How do you evaluate the counterfactual of an economy without QE programs? This, of course, is what differentiates Economics from Physics, where you can do controlled and repeated experiments.

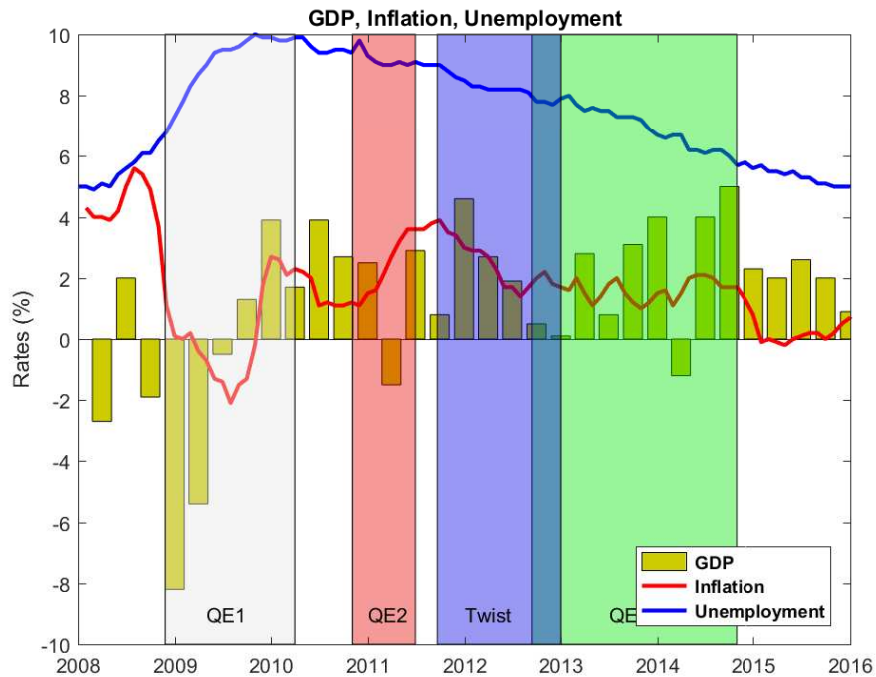


Figure 6: GDP, inflation, and unemployment rate since 2008.

At the time, two major concerns about the QE programs were hyperinflation and sharp dollar depreciation. As shown in Figure 6, the inflation rate has in fact been unusually low in recent years with the 2015 numbers hovering around 0. If you follow the currency market, you would know that in recent years the dollar has been strengthening against most currencies, and this is true even before the recent election). So what were the reasons? Let me quote Ben Bernanke again:

That idea (hyperinflation and sharp dollar depreciation) was linked to a perception that the Fed paid for securities by printing wheelbarrows of money. But contrary to what is sometimes said (and I said it once or twice myself, unfortunately, in oversimplified explanations), our policies did not in-

volve printing money – neither literally, when referring to cash, nor even metaphorically, when referring to other forms of money such as checking accounts. The amount of currency in circulation is determined by how much cash people want to hold (the demand goes up around Christmas shopping time, for example) and is not affected by the Fed’s securities purchases. Instead, the Fed pays for securities by creating reserves in the banking system. In a weak economy, like the one we were experiencing, those reserves simply lie fallow and they don’t serve as ‘money’ in the common sense of the word. As the economy strengthened, banks would begin to loan out their reserves, which would ultimately lead to the expansion of money and credit. Up to a point, that was exactly what we wanted to see. If growth in money and credit became excessive, it would eventually result in inflation, but we could avoid that by unwinding our easy-money policies at the appropriate time. And, as I had explained on many occasions, we had the tools we needed to raise rates and tighten monetary policy when needed. The fears of hyperinflation or a collapse of the dollar were consequently quite exaggerated. Market indicators of inflation expectations – including the fact that the U.S. government was able to borrow long-term at very low interest rates – showed that investors had great confidence in the Fed’s ability to keep inflation low. Our concern, if anything, was to get inflation a little higher, which was proving difficult to accomplish.

Finally, Figure 7 looks at the impact of the unconventional QE policy tools on the level as well as the slope of the yield curve. Again, causality is difficult to establish because we need to know the counterfactual of what would have happened if the Fed had not installed these policies. Also, the issue is further complicated by markets’ anticipations at the time as well as the endogeneity of the decision itself. All in all, however, these policy actions seem to be effective in keeping the long-term interest rate low.

- **Why So Much on QEs?** If you feel that I am writing too much here on quantitative easings (more than you need to know), I agree. But more information is always better than no information, right? Rest assured, I’ll not ask you to present the pro/con of the QE programs in the final exam.

I am recounting the events of 2010-2012 regarding quantitative easing for two reasons. First, these were really important events in the fixed income market. By going through the Fed’s balance sheet, you get a better sense as to how the Fed’s open market

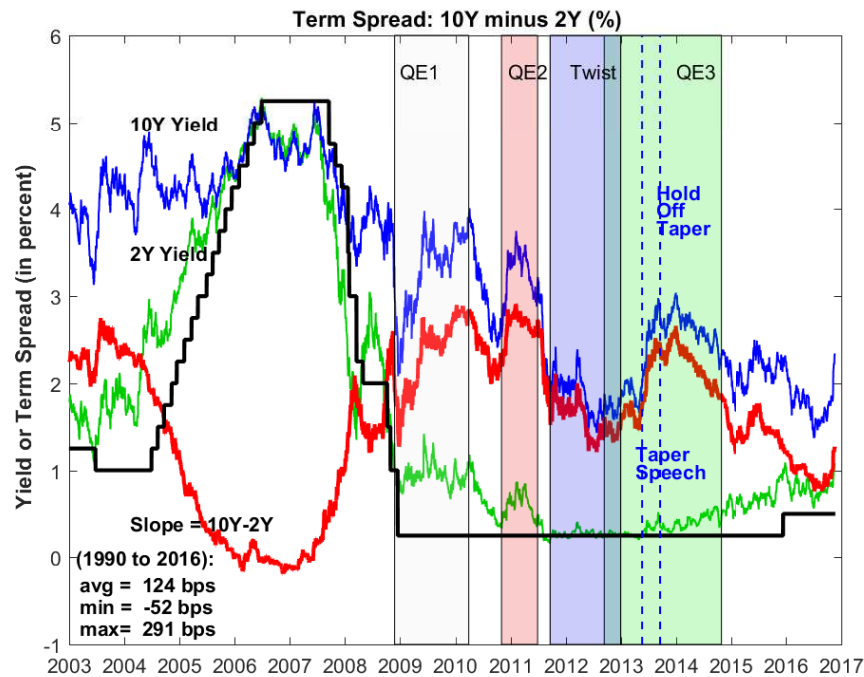


Figure 7: Treasury Yield Curve and Fed Fund Target Rate.

operation actually works. At least that was really helpful for me. The textbook information can be dry sometimes. Having plots like those in Figure 4 and 5 adds texture to my understanding. Second, I lived through that period in 2010-12 listening to many criticisms and derision against the QE programs. I am not a macro-economist and have not been trained in that field. And my thinking at the time was confused by many voices competing for attention. Personally, I find it is useful to read through the above two paragraphs written by Bernanke and look at the numbers for myself. So I thought I would share my readings with you.

I would not be surprised if, for each argument presented by Bernanke in his book, there is a counter argument. Honestly, the writings and thinking of some macro-economists are so complicated that they add more confusion than clarity. In my opinion, truth is always simple. It is the false that needs decoration. Complicated writing comes from a crowded and clouded mind. Unfortunately, in our field, complicated and convoluted thinking is often awarded with a premium because it is an exercise of a high IQ. In any case, for whatever it is worth, I appreciate the clear writing and thinking of chairman Bernanke.

2 Statistical Analysis of the Yield Curve

By now, we are comfortable with yield curves and have an intuitive understanding of the various factors influencing the short- and long-end of the yield curve. Let's now move on to quantify these random factors. Not surprisingly, the first risk factor that will show up through our analysis is the risk involved with duration. Second, we also noticed earlier that the entire yield curve does not move in tandem as in synchronized swimming. In particular, the long-end of the yield curve might not move entirely in parallel to the short-end of the yield curve. This points to the fact that the slope of the yield curve is not a constant. In fact, the slope becomes our second random factor. Finally, there might be some freedom in how the middle portion of the yield curve moves in relation to the short- and long-end. This observation gives rise to a third random factor called curvature.

It would not be surprising that market participants have long recognized the importance of these factors influencing the yield curve. But the concept of level, slope, and curvature was formally introduced in the 1991 paper by Litterman and Scheinkman, when both professors were working at Goldman Sachs. They identified these three common factors in the movements of yield curve through principal component analysis (PCA). In assignment 3, you get the chance to do this analysis yourself. The main difference is that their analysis is done in the yield space while your analysis will be done in the return space.

I'll go over this exercise in the yield space here in the notes.

- **Variance-Covariance Matrix:** Table 2 reproduces the content from Table 1 with the addition of 1Y yield. From Figure 1, we also notice that the 30Y yields were absent from February 19, 2002 to February 8, 2006 because the Treasury department suspended new issuance of 30-year bonds. In calculating the variance-covariance matrix, we will have to skip that specific period because of the missing 30-year bonds.

Table 2: Correlation and Standard Deviation of Daily Changes in Yields (1982 to 2015)

corr (%)	3M	1Y	2Y	5Y	10Y	30Y
3M	100.0	72.72	57.31	46.87	40.18	35.15
1Y	72.24	100.0	87.90	78.18	70.44	63.06
2Y	57.31	87.90	100.0	90.29	82.17	72.90
5Y	46.87	78.18	90.29	100.0	94.07	85.74
10Y	40.18	70.44	82.17	94.07	100.0	93.71
30Y	35.15	63.06	72.90	85.74	93.71	100.0
std (bps)	8.06	6.95	6.96	7.19	6.90	6.30

Let Cov be the variance-covariance matrix of the daily changes in yields for maturities 3M, 1Y, 2Y, 5Y, 10Y, and 30Y:

$$\text{Cov}(i, j) = \text{Corr}(i, j) \times \sigma_i \times \sigma_j ,$$

where σ is the standard deviation of the daily changes in yield.

- **Eigenvalue Decomposition:** Taking Cov as an input, we perform the eigenvalue decomposition. Let's first go through the calculations and then come back to understand what is really going on. The eigenvalue decomposition will give us two inter-related outputs. First, the eigenvalue E is a vector of six eigenvalues. This is because the dimension of the variance-covariance matrix is 6, one for each maturity. As shown in Table 3, we order the eigenvalues in the order of their magnitude. We call the eigenvalue with the largest magnitude PC1 (principal component one), the second PC2, and so on. The magnitudes of the eigenvalues might not be meaningful for you now, but it will be.

Table 3: Eigenvalues and Eigenvectors

Eigenvalues E	PC1	PC2	PC3	PC4	PC5	PC6
E (bps ²)	226.99	50.14	13.77	5.45	2.86	1.47
E/sum(E) (%)	75.49	16.68	4.58	1.81	0.95	0.49

Eigenvectors D	PC1	PC2	PC3	PC4	PC5	PC6
3M	0.3630	-0.8017	0.4347	0.1876	-0.0365	0.0006
1Y	0.4182	-0.2371	-0.4682	-0.6806	0.2939	0.0016
2Y	0.4351	0.0257	-0.5134	0.3309	-0.6505	0.1176
5Y	0.4513	0.2493	-0.0709	0.4572	0.5076	-0.5124
10Y	0.4176	0.3430	0.2837	0.0418	0.2271	0.7577
30Y	0.3550	0.3472	0.4926	-0.4258	-0.4242	-0.3866

Second, associated with each eigenvalue is a vector, called eigenvector. There are six eigenvalues. So there are six eigenvectors, one for each eigenvalue. Putting these six eigenvectors together, we have a matrix D that is 6 by 6, as shown in Table 3. Let's now go over the first three PCs:

- **Level:** As shown in Table 3, associated with PC1 is the first eigenvector:

$$D^{PC1} = \begin{pmatrix} 0.3630 \\ 0.4182 \\ 0.4351 \\ 0.4513 \\ 0.4176 \\ 0.3550 \end{pmatrix},$$

which is a vector of six, one for each maturity. So effectively, the first PC is close to an equal-weighted portfolio of all six yields (or daily changes in yields, to be more precise). This factor corresponds to a movement in the yield curve when all six yields move up and down in tandem or in parallel. In other words, it captures the level movement and the best measure for exposure to this level risk is duration.

- **Slope:** Associated with PC2 is the second eigenvector:

$$D^{PC2} = \begin{pmatrix} -0.8017 \\ -0.2371 \\ 0.0257 \\ 0.2493 \\ 0.3430 \\ 0.3472 \end{pmatrix},$$

which is a long/short portfolio along the maturity dimension. It is long long-term yield and short short-term yield. It really does not matter which end of the yield curve is being long, as long as the weights on the long-end are opposite to the weights on the short-end. Naturally, you think “slope.”

- **Curvature:** Associated with PC3 is the third eigenvector:

$$D^{PC3} = \begin{pmatrix} 0.4347 \\ -0.4682 \\ -0.5134 \\ -0.0709 \\ 0.2837 \\ 0.4926 \end{pmatrix},$$

which is again a long/short portfolio along the maturity dimension, but it is long both short- and long-end of the yield curve, and short the middle part of the yield curve. Again, the exact sign of long/short does not really matter as long as the weights on the short- and long-end are opposite to the weights on the middle portion of the yield curve. So this reason, this factor is called “curvature.”

Figure 8 summarizes the first three PCs in a plot, which might be more intuitive for us to see the meaning of level, slope, and curvature.

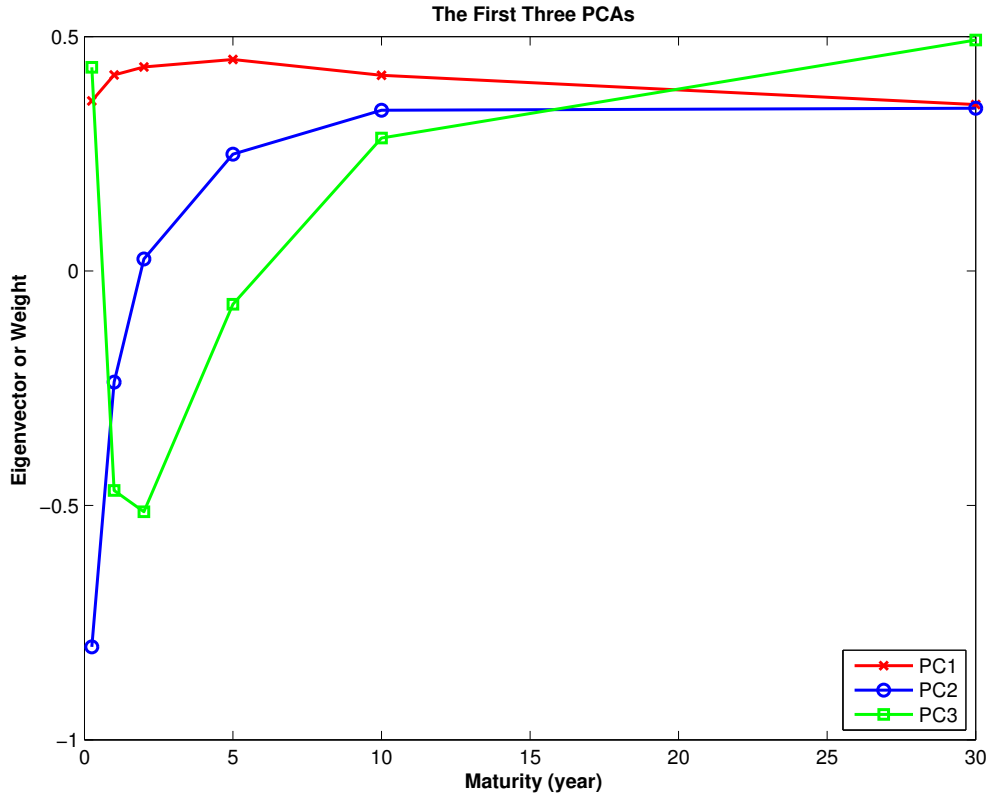


Figure 8: Level, Slope, and Curvature.

- Relative Importance of the PCs:** We focus on the first three PCs because of their relative importance. To see this, let's go back to the eigenvalues in Table 3. By construction, the eigenvalue associated with PC1 is the highest in magnitude. Let's now construct a time-series of PC1 using the weights subscribed in D^{PC1} (avoiding the 2002-2006 period because of the missing 30-year yields). The standard deviation of this portfolio turns out to be 15.07 bps, and the variance is ... 226.98 (bps²). You can repeat the same exercise for all other PCs. In short, the n -th eigenvalue is in fact the variance of the n -th PC.

What is cool about the eigenvalue decomposition is that it transforms the original data (with correlated yields) into six independent random factors: PC1, PC2, etc. (Please double check this statement by constructing time-series of PC1 and PC2 and calculate their correlation.) As a result, working with mutually independent PCs is more convenient than working with correlated yields. Because all six factors are independent, we can add now all six eigenvalues into $\text{sum}(E)$ and use it as a normalizing factor for E . As shown in Table 3, the first PC accounts for 75.49% of the total variance, the second PC accounts for 16.68%, and the third PC accounts for 4.58%. Adding all three together, we see that they account for 96.75% of the total variance. This is why most of the term structure models use three factors. This is also why duration hedging, which is a hedge against PC1, is the most important form of hedging in the fixed income market.

Once a portfolio is hedged with zero duration, then the slope exposure becomes the most important risk. Once a portfolio is hedged with duration and slope, then you worry about curvature exposure. In the old days, there are butterfly trades which are duration and slope neutral, and are structured so that the main exposure is the curvature risk. Of course, you need to be a fixed-income nerd to get this deep into the yield curve trades.

- **More on the Eigenvectors D:** By now, we understand that there are six eigenvectors and putting them together gives us a 6×6 matrix. Each eigenvector is a vector of portfolio weights (not normalized) on the six maturities.

Let's now take a closer look. Let's start with the important observation that all six PCs are independent. So pick any two PCs, say PC1 and PC2, and their correlation will be zero. As mentioned earlier, this is why eigenvalue decomposition is useful. It gives us independent factors. Let's use the matrix notation for the following calculations. First we know that

$$PC1_t = (D^{PC1})^\top \times \Delta y_t,$$

where Δy_t is the vector of daily changes in yields for the six maturities and $(D^{PC1})^\top$ is the transpose of D^{PC1} . Of course, we also know that

$$PC2_t = (D^{PC2})^\top \times \Delta y_t.$$

So if $\text{cov}(PC1_t, PC2_t) = 0$, then it must be that

$$(D^{PC1})^\top \times D^{PC2} = 0.$$

Applying this logic pairwise to all maturities, you will be convinced that

$$D^\top D = I ,$$

where I is an identity matrix of dimension 6×6 , with diagonal terms equaling 1 and off-diagonal terms equaling zero. In other words,

$$D^{-1} = D^\top .$$

If you don't believe me, just try it out using Excel or Matlab.

- **Running Regressions:** Now let's take a look at Table 4, where I report the following regression results:

$$\Delta y_t = a + \beta^{PC1} PC1_t + \beta^{PC2} PC2_t + \beta^{PC3} PC3_t + \epsilon_t .$$

Knowing that all three PCs are independent, we can calculate the individual R-squared for each PC and add them together to get the total R-squared of the regression.

Table 4: Regressing Δy on the First Three PCs

maturity	PC1 β	PC2 β	PC3 β	PC1 R2 (%)	PC2 R2 (%)	PC3 R2 (%)	Total R2 (%)
3M	0.3630	-0.8017	0.4347	46.06	49.63	4.01	99.70
1Y	0.4182	-0.2371	-0.4682	82.18	5.83	6.25	94.26
2Y	0.4351	0.0257	-0.5134	88.67	0.07	7.49	96.23
5Y	0.4513	0.2493	-0.0709	89.46	6.03	0.13	95.62
10Y	0.4176	0.3430	0.2837	83.17	12.39	2.33	97.89
30Y	0.3550	0.3472	0.4926	72.04	15.22	8.41	95.66

First, you can see that PC1 remains the most important random factor, explaining the daily changes in yields with very high R-squared's. For the two extreme ends of the yield curve (3M and 30Y), the explanatory power is relatively weaker. This is where PC2 picks up. In particular, PC2 contributes quite a bit in explaining the movements in the short-end of the yield curve. Adding all three PC factors, we can explain the random variations in daily changes in yields with R-squared's that are well above 90%.

Second, take a look at the regression coefficients β 's. What do you see? Compare D^{PC1} with the beta coefficients on PC1, there are identical! Likewise for D^{PC2} and

β^{PC2} , and D^{PC3} and β^{PC3} . Can you prove this result mathematically? (No worries, I'll not ask you to do this proof in the exam.)

- **More on PCA:** What we've talked about so far in this section is statistical based. The yield curve is well suited for a statistical analysis like PCA. Once you understand the mechanics of the PCA, it will be instructive for you to go back to the economic drivers for these common risk factors in the fixed-income market.

More broadly, the PCA approach can also be used in many markets where the observables are correlated due to some common factors. For example, applying PCA to international equity returns, the first PC will be a world index with roughly equal weight on all countries. The second PC will be a long/short portfolio across the two most representative regions (which could change over time).

Whatever you might do with PCA, just be reminded that this is simply a statistical tool that helps you extract mutually independent factors, and the importance of the factors are ordered by their variances (i.e., eigenvalues). Also remember that the only input for the eigenvalue decomposition is the variance-covariance matrix. Use this tool effectively for the your desired objective. All all is done, take the extra step to understand the economic and institutional drivers for the extracted factors.