Jun Pan
junpan@mit.edu

MIT Sloan School of Management
15.433-15.4331 Financial Markets, Fall 2016

253-3083
E62-624

Class 19: Fixed Income, Yield and Duration

This Version: November 28, 2016[1]

1 From Equity to Fixed Income

- **Vehicles for Risk:** Moving from equity to options to bonds and, later, to OTC derivatives, there is always one thing in common: each market is a vehicle for risk. The nature and origin of the risk might vary from one market to the other, but our approach to risk remains the same.

  We plot the time-series data to see how it varies over time. We map the historical experiences into a distribution and use it as a basis to envision future scenarios. Thinking of the future in a static fashion as one fixed future date, we employ random variables to model the distribution at this future date (e.g., the CAPM). Thinking of the future in a dynamic fashion as a path leading into the future, we use stochastic processes to model the random paths (e.g., Black-Scholes). Either way, we use these models to price the risk involved, taking into account not only the likelihood and magnitude of the risk, but also investors' attitudes to the risk. After this is done, we go back to the data to see how well our model performs. Very often, the data surprises us. In this process of model meeting the data, new insights arise.

- **Relating one to the other:** You might also notice that, in Finance, we keep ourselves busy by relating one thing to the other. For example, in the equity market, we relate the individual stock returns R_t^i to the contemporaneous returns of the market portfolio R_t^M. The pricing of an individual stock is done through the pricing of the market:

$$E(R_t^i) - r_f = \beta^i \left(E(R_t^M) - r_f \right) .$$

[1] This note was originally written in November 2015. I have not had the chance to update it for Fall 2016. In many places, "right now" means Fall 2015. Just a quick update on the numbers: as of November 16, 2016, the three-month Treasury yield is at 46 basis points, the 10-year yield is at 2.22%, and the 30-year at 2.92%. On November 8, 2016, the 10-year was at 1.88%, followed by 2.07%, 2.15%, and 2.23% on November 9, 10, and 14.

By doing so, we narrow our attention down to one risk factor: the market portfolio. In the crowd of thousands of stocks, your eyes are on this one and one thing only, and everything else fades into the background.

In options, we relate the time-t option price C_t to two things: the price of the underlying stock S_t and the volatility of the underlying stock σ. The relation between C_t and S_t is useful, but what really makes options unique is the relation between C_t and σ. This is especially important when we step outside of the Black-Scholes model and allow σ_t to vary over time: now options are unique vehicles for the risk in σ_t. This is why I asked you to pay special attention to this approximation for an ATM option:

$$C_t/S_t = P_t/S_t \approx \frac{1}{\sqrt{2\pi}}\,\sigma\sqrt{T}\,.$$

Now we are studying the fixed-income market, which is large and important, encompassing products such as Treasury bonds (\$12.5tn), mortgage-backed securities (\$8.7tn), corporate bonds (\$7.8tn), Muni (\$3.6tn), money market funds (\$2.9tn), agency bonds (\$2.0tn), and asset-backed securities (\$1.3tn). The numbers in parentheses are amount outstanding as of end 2014. At the center of our attention is the risk that is common to all of these products: interest rate fluctuations. Not one interest rate, but many: one for each maturity. Putting them together, we have a yield curve. In Finance, there is no other risk that is more important than this yield curve risk. It is fundamental to everything we do in Finance. It is the basis from which all other discount rates are calculated.

In dealing with this risk, we prefer to work in the yield space because it is more convenient, but the profit/loss happens in the dollar space. As a result, we will be busy relating one thing to the other again. This gives rise to concepts such as duration and convexity. An outsider might look at these funny names and accuse people in Finance of creating unnecessary concepts so as to confuse and take advantage of those who know less about finance. There might be such practices going on elsewhere on Wall Street, but concepts such as duration and convexity and Black-Scholes implied vol are created out of necessity. I cannot imagine myself navigating the bond market without having tools like duration and convexity.

- **Focus on What's Important:** In talking about beta in equity, implied-vol in options, and duration and convexity in bonds, my intention is to remind you to focus on what's important.

Often, I notice that some students have the tendency to focus on the small and trifling things first before trying to digest the more important message. When you look at a tree, your attention goes first to the overall structure and shape, not to a small offshoot from a branch of the tree (unless there is a cat sitting there). If you are drowning, you grab the nearest and largest lifesaver available; you don't stop to examine the color or the make of the lifesaver. Nor do you question whether or not the lifesaver is made of sustainable materials.

So please, go for the important concept first. Only after you understand these concepts really well, then you have the luxury in digging into the minute details. Of course, ideally, you would like to be good at both: big-picture and rigor. But in the process of learning, it makes sense to go after the big picture first.

While I am on this topic, let me also add that you should always bring your common sense back to anything you do in Finance. For example, it is very easy to get lost when working on a project. Sooner or later, the model and the spreadsheet become the boss and you the slave. Use your common sense. Don't invest in any fancy models or techniques until you have a very clear view of why you need them. Otherwise, it will be garbage in and garbage out. In the process, you might manage to impress yourself and a few others with the fancy techniques and models. But in truth, it is mostly confusion.

The same thing applies to a professor. If, after each class, I make you more confused than before, then I am not doing a good job in teaching the materials. That is why I am writing the lecture notes, to give myself ... a second chance.

- **In the Return Space:** Coming back to our main topic, I list in Table 1 summary statistics of equity (the CRSP value-weighted index) and bond returns using monthly data from 1942 through 2014. In the second panel of the table, I also report the numbers for the more recent period from 1990 through 2014.

  For the sample period from 1942 through 2014, the average monthly return of the US stock market is 1.03% and the volatility is about 4.16%. In annualized terms, the average return is 12.33% and the volatility is 14.4%. (The 20% annual volatility number we've been using includes the great depression.) For the same period, the average return of a 10-year bond is about 47 basis points per month and the volatility is about 2%. Not surprisingly, with decreasing maturity (and duration), both the average return and volatility decrease for shorter maturity bonds. The one-month TBill has an average return of 32 basis points per month, and an average yield of

$0.32\% \times 12 = 3.84\%$. The monthly volatility of the one-month Treasury bill is 0.26%, which is only a small fraction of that in the stock market (4.16%).

Table 1: Monthly Equity Returns and Bond Returns

Monthly 1942-2014	mean (%)	std (%)	Sharpe ratio	min (%)	max (%)	correlation with Stock	TBill	10Y
Stock	1.03	4.16	0.17	-21.58	16.81	1.00	-0.05	0.10
10Y Bond	0.47	2.00	0.08	-6.68	10.00	0.10	0.12	1.00
5Y Bond	0.46	1.38	0.10	-5.80	10.61	0.07	0.19	0.90
2Y Bond	0.42	0.77	0.13	-3.69	8.42	0.08	0.37	0.76
1Y Bond	0.40	0.50	0.16	-1.72	5.61	0.08	0.59	0.62
1M TBill	0.32	0.26		-0.00	1.52	-0.05	1.00	0.12
CPI	0.31	0.45		-1.92	5.88	-0.07	0.26	-0.07
Monthly 1990-2014	mean (%)	std (%)	Sharpe ratio	min (%)	max (%)	correlation with Stock	TBill	10Y
Stock	0.87	4.22	0.15	-16.70	11.41	1.00	0.01	-0.06
10Y Bond	0.57	1.99	0.16	-6.68	8.54	-0.06	0.07	1.00
5Y Bond	0.50	1.24	0.20	-3.38	4.52	-0.10	0.15	0.93
2Y Bond	0.39	0.54	0.26	-1.30	2.07	-0.11	0.41	0.74
1Y Bond	0.33	0.31	0.26	-0.33	1.31	-0.03	0.72	0.51
1M TBill	0.25	0.19		-0.00	0.68	0.01	1.00	0.07
CPI	0.21	0.34		-1.92	1.22	-0.04	0.18	-0.16

Table 1 also reports the best and worst one-month returns for each of the securities. Not surprisingly, the stock market is the most risky with the largest range of minimum and maximum. During the sample period from 1942 to 2014, the worst one-month return was -21.58%, which happened in October 1987.

Also reported are the correlations between the stock returns and the bond returns. The correlation between these two markets is very weak and is also unstable. The correlation between stock and 10-year bond is 10% for the sample from 1942 through 2014 and -6% for the more recent sample from 1990 through 2014. Unlike the low correlation between stock and bond, the correlations between the bond returns are relatively high. The closer the maturity (e.g., 10Y and 5Y), the higher the correlation. We will come back and the investigate this issue in our next class when we do PCA (Principal Component Analysis) on bonds.

It is also interesting to see that the correlations between inflation (CPI) and the stock returns and 10Y are low and slightly negative. The correlation between inflation and the 1M Tbill is about 26% for the entire sample and 18% for the more recent sample. Note that we are working with nominal interest rate, which is the sum of real interest rate and inflation. As you can see from Table 1, the average inflation is close to the 1-month Treasury bill, but slightly lower, implying that the real interest rate is on average positive.

- **The Cycle of Hot and Cold:** Using the average return of the one-month Treasury bill as the riskfree rate, we can calculate the Sharpe ratios of the equity and bond returns. From this perspective, bonds have been more attractive (higher average return and lower volatility) for the more recent sample period from 1990 through 2014.

  In fact, from the mid 1980s to today, the bond market condition has been quite favorable. The interest rates have been decreasing from the double digits in the early 1980s to today's near-zero. Some call it a 30-year bull market run. In addition to the favorable market condition, we have also seen the rise of MBS, junk bonds, OTC derivatives, asset-backed securities, all of which add to the business of fixed-income desks in investment banks.

  When Michael Lewis joined the training program in Salomon in 1985, the bond market was just getting hot, driven by the profitability in bonds. In 1986, other firms like Goldman Sachs were catching up with Salomon's bond expertise by hiring people away from Salomon (See, for example, *Money and Power* by Cohan). Within Salomon, as described in Michael Lewis' book, *Liar's Poker*, an entertaining (maybe too entertaining) book, the desired location was to be on a bond desk. Equity was looked down up, and "Equity in Dallas" was the equivalent of Siberia.

  But only ten years prior to that, bond was not at hot and equity was the place to go. Quoting Michael Lewis,

  > That, anyway, is what I was told. It was hard to prove any of it because the only evidence was oral. But consider the kickoff chuckle to a speech given to the Wharton School in March 1977 by Sidney Homer of Salomon Brothers, the leading bond analyst on Wall Street from the mid-1940s right through to the late 1970s. "I felt frustrated," said Homer about his job. "At cocktail parties lovely ladies would corner me and ask my opinion of the market, but alas, when they learned I was a bond man, they would quietly drift away."
  >
  > Or consider the very lack of evidence itself. There are 287 books about bonds in the New York Public Library, and most of them are about chemistry. The ones that aren't contain lots of ugly numbers and bear titles such as *All Quiet*

> *on the Bond Front*, and *Low-Risk Strategies for the Investor*. In other words, they aren't the sort of page turners that moisten your palms and glue you to your seat. People who believe themselves of social consequence tend to leave more of a paper trail, in the form of memoirs and anecdotiana. But while there are dozens of anecdotes and several memoirs from the stock markets, the bond markets are officially silent. Bond people pose the same problem to a cultural anthropologist as a nonliterate tribe deep in the Amazon.

By now, bond people are certainly not the equivalent of a nonliterate tribe deep in the Amazon. In fact, if you search Amazon for books on Finance, many of them were written by bond traders. So is this endless cycle of being hot and cold, in and out of favor. Whatever that can go up certainly has the potential to come down. The moment something is in favor marks the beginning of its decline.

Right now (Fall 2015), the interest rate is at a level as low as it can ever be, and the 30-year bull run in the bond market is approaching to an end. Most likely, the Fed will raise the Fed fund rate in its December FOMC meeting this year (Fall 2015). Inferring from the pricing in Fed fund futures, there is a 70% likelihood of a Fed hike at its December 15-16 meeting (Fall 2015). So we will know the result before our final exam on December 17 (Fall 2015).

In the mutual fund world, the famous bond fund, Pimco's Total Return, is a good representation of this cycle of bull and impending bear. As shown in Figure 1, the first observation of Pimco (Total Return Fund, Institutional Class) in my data was at the end of June 1987 with a total net asset value of $12.8 million. From 1987 to 2013, the fund, benefited from the favorable bond market condition, was in a steady ascend, reaching to its peak ($182.8 billion) in April 2013. This grow in the size of a mutual fund has two component: the market performance and fund flows. So the growth from $18 million to $182.8 billion was a combination of both. As we know, in the mutual fund word, flow chases performance. So the favorable condition in the bond market has a lot to do with the growth.

In recent months, the size of the fund has been decreasing quite rapidly. Figure 2 plots the total net asset value for all four classes of the fund. Of course, if you have been following the news since 2014, you would know that the internal powerful struggle and the clash of personalities also contributed to the fund outflow. But the clash of personality probably would not have escalated to such a degree had the bond market condition been favorable.

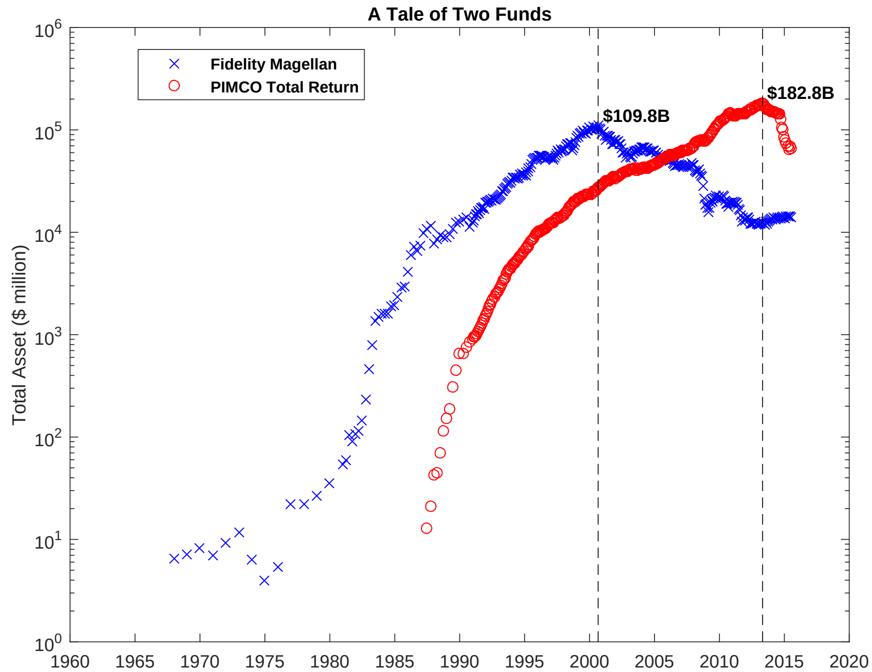

Figure 1: Total Net Asset Value, Fidelity Magellan and Pimco Total Return.

Also note that the plot is in log-scale, in an effort to damp the high growth rate. If it were plotted in a linear plot, the ups and downs would have been even more dramatic.

As another example of the force of the overall market condition versus the skills of an individual fund manager, I plot in Figure 1 the total net asset value of the once famous equity fund, Fidelity Magellan. The fund shows up in my data since May 1963 but the first reported total net asset value in my data was $6.5 million in December 1967. By December 1975, the fund was smaller at $5.4 million, most likely due to the bear market of 1973-74. In June 1976 Peter Lynch took over the fund. From 1976 to 1990, under Peter Lynch's management, the fund grew in size as well as in fame. After Peter Lynch's retirement in May 1990, the fund kept growing, thankful to the bull market of the late 1990s. The fund grew to its peak ($109.8 billion) in August 2000, and then started its decline after the Internet bubble burst. Right now (Fall 2015), it is a $14 billion fund, roughly the size when Peter Lynch retired from the fund in May 1990.

Cycles like those in Figure 1 are part and parcel of the financial markets. Such forces in financial markets should be humbling for any human being, no matter how successful this person might be. To attribute one's success entirely to one's talent is pure arrogance and ignorance. If you have not read the recent stories surrounding Bill Gross (the co-founder of Pimco), I would suggest that you do. At some point in your life,

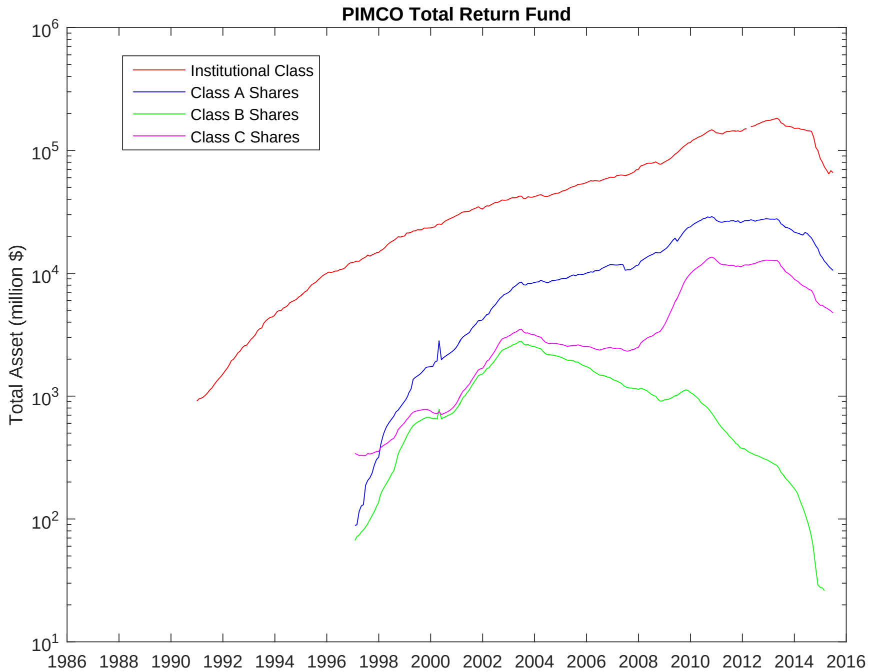

Figure 2: Total Net Asset Value, Pimco Total Return Fund.

you might get lucky and become successful. Try not to let your ego drive you too far. There are no worse enemies in your life than your own ego. In fact, your ego is you only enemy.

2 Bond Price and Yield: Duration and Convexity

- **Bond Price P and Yield to Maturity y:** A Treasury yield curve involves Treasury bonds, notes, and bills. Treasury notes are issued in terms of 2, 3, 5, 7, and 10 years; Treasury bonds are issued at 30 years. A Treasury bond issued 25 years ago would have 5 years to maturity, same as a newly issued 5-year notes. But the coupon rates of the two bonds are different. Coupon bearing bonds are issued at par, making the coupon rate close to the yield to maturity at the time of issuance. Given the current low interest rate environment, the 30-year bond issued 25 years ago has a coupon rate that is higher than the newly issued 5 year notes. It is therefore a premium bond. There are also differences in liquidity, which we will talk about later.

  Throughout the fixed-income classes, I'll not make a distinction between notes and bonds and will refer to them simply as bonds. I'll use the notation of P_t as the bond price at time t, and $y_t\%$ as the yield to maturity at time t. At issuance, a Treasury bond is defined by the following parameters: **face value** = \$100; **coupon rate** = c; **maturity** = T years. These parameters are fixed throughout the life of the bond and will not change. Treasury bonds pay coupon semi-annually, and, at issuance, the coupon rate c is chosen so that the bond is priced at par with $P = \$100$. As a result, the yield to maturity y (semi-annual compounding) equals to the coupon rate c when the bond was first issued.

Later, with the fluctuations in interest rates, both P and y will change. There is a deterministic relation between the two:

$$P = \sum_{n=1}^{2T} \frac{\frac{c}{2} \times 100}{\left(1 + \frac{y}{2}\right)^n} + \frac{100}{\left(1 + \frac{y}{2}\right)^{2T}}, \tag{1}$$

where both c and y are expressed in percentage. So an increasing interest rate environment after the issuance of the bond is bad news for long-only bond investors: P decreases with increasing y and the bond will be in discount ($P < \$100$). Conversely, a decreasing interest rate environment is good news such a long-only bond investor: P increases with decreasing y and the bond is in premium ($P > \$100$).

So Treasury bonds are not at all riskfree, and its volatility is driven by the volatility of the interest rate. Assuming the high credit quality of the US government, the Treasury bonds are considered to be almost default free. During the heat of the debt-ceiling crisis in 2011, the rating agency S&P downgraded the US Treasury from AAA to AA+. The financial markets were in a crisis mode and Treasury bonds actually appreciated in value because, out of the flight to quality, investors move their capital away from risky assets to ... the US Treasury bonds.

The relation between P and y as expressed in Equation (1) is a very important one, and we will come back to it again. For now, I would like you to keep the picture of Figure 3 in mind. This is what the payoff schedule of a bond looks like. Over the life of the bond, you collect small coupon payments every six months, and toward the end of the life of the bond, at maturity, you collect the last coupon payment plus the principal. You discount this cashflow by a constant interest rate y using the discount function $1/(1 + y/2)^n$ for the n-th semi-annual payment. In doing this calculation, you link the bond price P to its yield to maturity y. There is no uncertainty involved in this relationship. There is also no economics involved in this calculation. But the calculation becomes very handy as we move between P and y. Concepts such as duration and convexity arise out of this calculation.

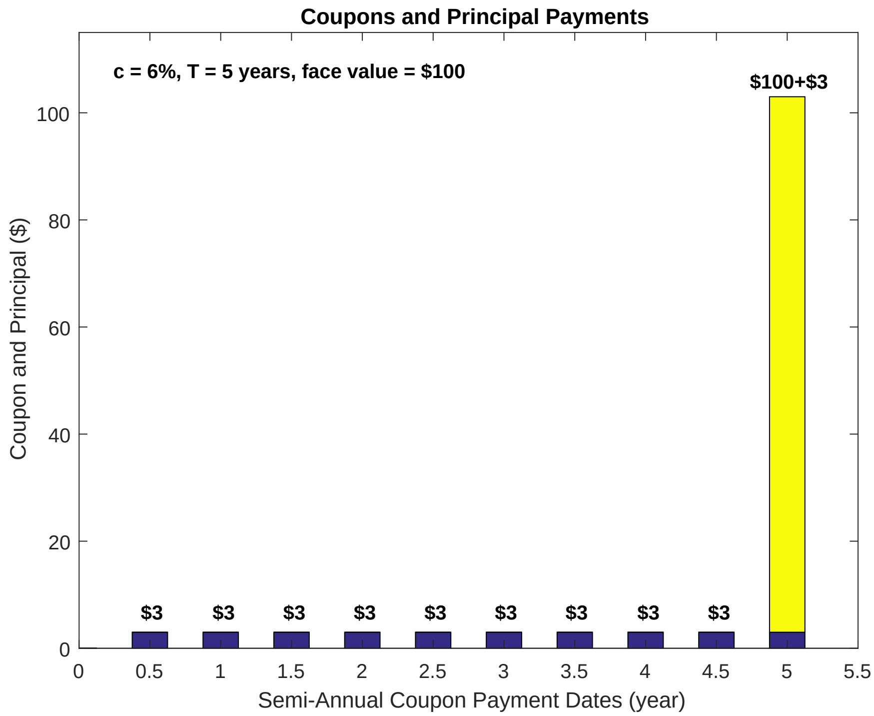

Figure 3: Coupon and Principal Payment Dates

- **Treasury Yield Curve:** As shown in Figure 4, a Treasury yield curve is plot of yield against maturity, for Treasury bonds of varying maturities. Treasury bonds are traded in terms of market prices P. So a yield curve is constructed using the market prices of individual Treasury bonds. In Figure 4, the green dots are Treasury bills, the blue dots are Treasury notes, and the purple dots are old Treasury bonds. For example, the yield curve in Figure 4 was plotted for November 8, 1994. For a purple dot with a maturity of seven years, the bond was issued 23 years ago in 1971 as a 30-year Treasury bond.

  As you can see, the yield curve is not created in vacuum. It is made up of individual bonds. In fact, the creation of a yield curve is not a simple task. The various bonds have different liquidity: the old bonds are typically less liquid while the new bonds/notes are typically very liquid. The liquidity effect shows up in the market prices of these bonds: illiquid bonds are cheaper than the liquid bonds. As a result, in constructing the yield curve, considerations such as liquidity take place. I do not want to make you a specialist in curve fitting, but if we have time in the next class, I will talk more about curve fitting.

  Focusing back on the yield curve in Figure 4, we see that on this day, the term structure is upward sloping. The short end of the yield curve is about 4.6%, the 2-year yield

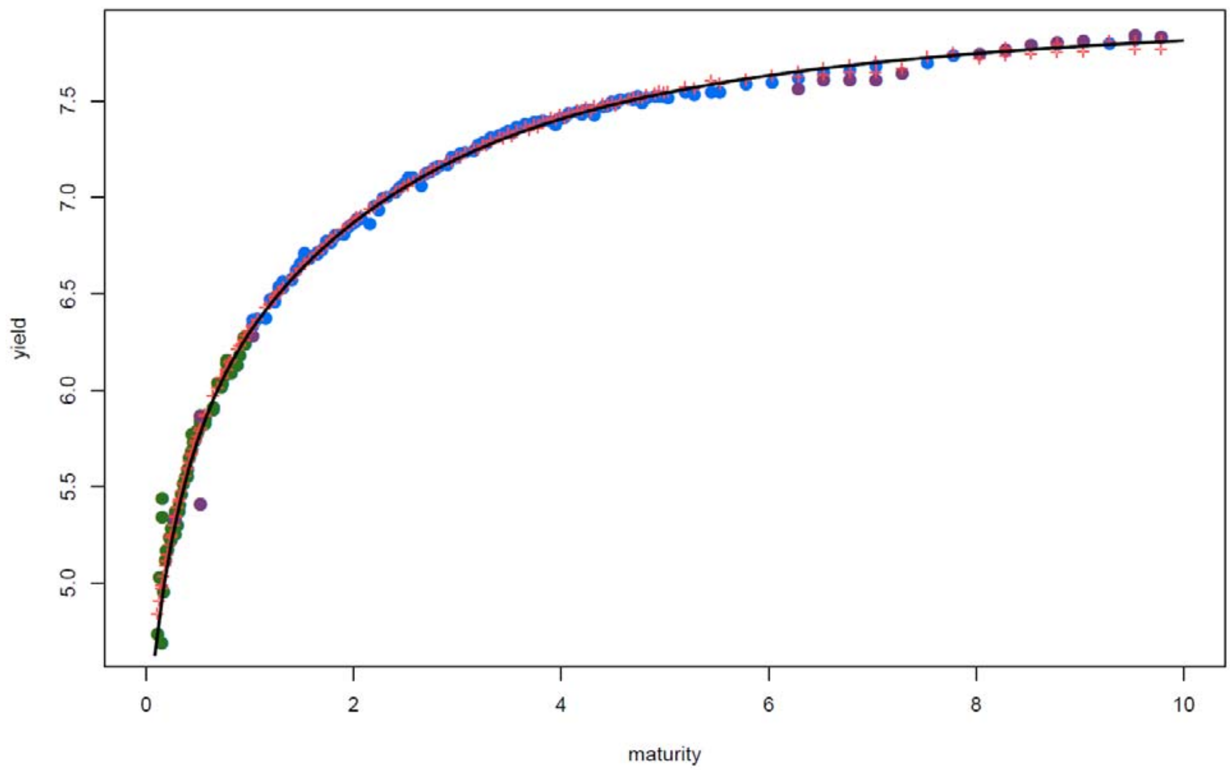

Figure 4: Treasury Yield Curve on November 8, 1994.

is about 6.8%, and the 10-year yield is at 7.8%. This makes the 10y to 2y spread at about 100 basis points. For bonds of similar maturities, the spreads are quite tight, indicating active arbitrage activities on the yield curve. By comparison, the yield curve on December 11, 2008, plotted in Figure 5, looks quite dramatic. Bonds are very similar maturities are trading at a yield spread in the order of 50 basis points. During normal market conditions, spreads so wide would never happen in this market. Of course, December 2008 was not normal. This picture indicates the lack of arbitrage activities in 2008, even in the most liquid market.

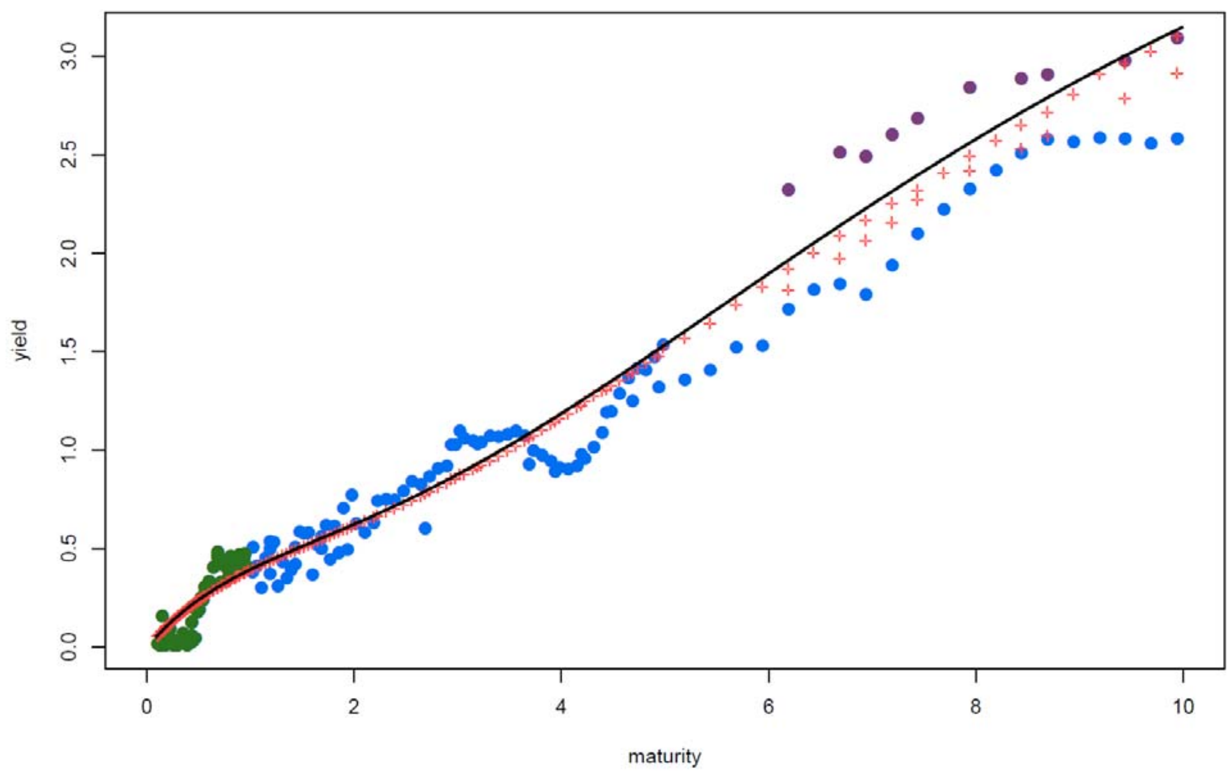

Figure 5: Treasury Yield Curve on December 11, 2008.

- **Time-Varying Yields:** To understand how the yield curve move over time, Figure 6 plots the time-series of Treasury constant maturity yields for a few selected maturities.

  These constant maturity yields are calculated daily by using market prices of Treasury bonds as the input. And the output is the par-coupon yields of varying maturities. Effectively, these are interpolated yields for the a set of fixed maturity of interest (e.g., 1, 2, 3, 5, 7, 10, 20, and 30 years). Again, to know what is really going on, we need to

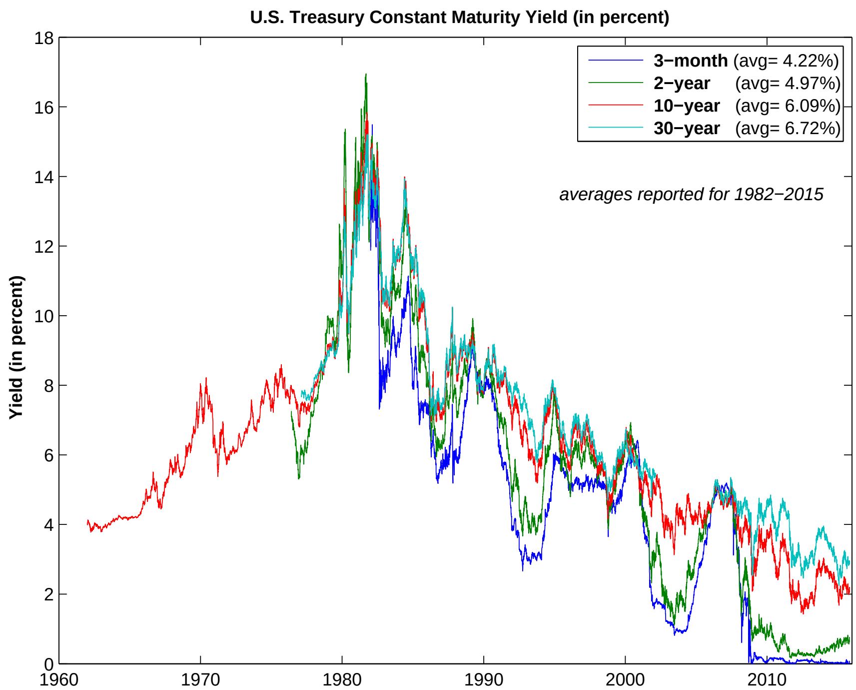

Figure 6: Time-Series of Treasury Constant Maturity Yields.

spend some time on curve fitting. For those who are interested, this is a not so useful explanation from the Treasury department, but it is better than nothing.

Let's now used these CMT yields and see how the yield curve vary over time. As shown in Figure 6, most of the time, the yield curve is upward sloping. Using data from 1982 to today, the 2-year CMT yield is on average 4.97%, the 10-year yield is on average 6.09%, and the 30-year yield is on average 6.72%. So the spread of 10y to 2y is on average 100 basis points. There are also times when the yield curve is not so steep or even inverting. We will take a closer look later on these events. Also notice that the green line (2yr yield) is picking up in recent days. The 2yr yield is a policy sensitive yield and is moving up in anticipation of a rate hike.

Also notice the missing 30yr yield in Figure 6 from early 2002 to early 2006. In late 2001, facing projections of burgeoning surpluses, the Treasury decided to stop issuing the 30-year bond to save tax payers money. In late 2005, the Treasury decided to re-introduce the 30-year bond and held its first auction in fives years on February 9, 2006.

Using these CMT yields, let's also calculate the daily volatility of the Treasury yields. As shown in Table 2, using daily data from 1982 to today, the standard deviation of the daily changes in the 3M Tbill rate is about 7.63 basis points. The 2Y and 10Y yields are slightly less volatile, at around 6.8 basis points. In recent years, however, the volatility is low for the short end because of the monetary policy. In general, however, the short end of the yield is typically more volatile, although the different in volatility is not huge. In other words, when measured in the yield space, the volatility across different maturity is comparable. But when it comes to the return space, the volatility across different maturity will be very different because of the difference in duration, which we will see shortly.

Table 2: Summary Statistics of Daily Changes in Treasury Yields

sample	maturity	std (bp)	min (bp)	date	max (bp)	date
1982-2015	3M	7.63	-104	19820222	169	19820201
	2Y	6.86	-84	19871020	80	19820201
	10Y	6.80	-75	19871020	44	19820201
	30Y	6.30	-76	19871020	42	19820201
1990-2008	3M	5.18	-64	20070820	58	20001226
	2Y	6.05	-54	20010913	36	19940404
	10Y	5.78	-23	19950613	39	19940404
	30Y	4.99	-33	20011031	32	19940404
2008-2015	3M	4.94	-81	20080917	76	20080919
	2Y	4.86	-45	20080915	38	20080919
	10Y	6.42	-51	20090318	24	20080930
	30Y	6.12	-32	20081120	28	20110811

Table 2 also reports the largest one day movements for these yields. Let me link a few of these extreme movements in yield to the events at the time:

– October 20, 1987 was the day after the 1987 stock market crash.

– April 1994 was a very testy time in the bond market because of monetary policy tightening by Chairman Greenspan.

– September 15 to 19, 2008 was the week of Lehman default and AIG bailout. TBill rates first decreased sharply (increased in value) because of flight to quality and then bounced back on September 19.

– On March 18, 2009, the Fed made the following announcements, which were summarized in Chairman Ben Bernanke's recent book. *The overall package was designed to get markets' attention, and it did. We announced that we planned to* *increase our 2009 purchases of mortgage-backed securities guaranteed by Fannie, Freddie, and Ginnie Mae to \$1.25 trillion, an increase of \$750 billion. We also doubled, from \$100 billion to \$200 billion, our planned purchases of the debt issued by Fannie and Freddie to finance their own holdings. We would also buy \$300 billion of Treasuries over the next six months, our first foray into Treasury purchases. Finally, we strengthened our guidance about our plans for our benchmark interest rate, the federal funds rate. In January, we had said that we expected the funds rate to be at exceptionally low levels "for some time." In March, "for some time" became "for an extended period." We hoped that this new signal on short-term rates would help bring down long-term rates.*

– The across-the-board increase in yield on February 1, 1982 was likely caused by the monetary policy tightening under Chairman Paul Volcker.

Overall, the numbers presented in Table 2 give us a baseline in observing and judging the daily movements in interest rates. A one-sigma move in this market is about 6 to 7 basis points. A daily movement of 25 basis points is unusual for this market.

- **Dollar Duration:** There are two measures of duration that is important for us to know. The dollar duration is defined as

$$-\frac{\partial P}{\partial y} = \frac{1}{1+\frac{y}{2}}\left[\sum_{n=1}^{2T} \frac{n}{2} \times \frac{\frac{c}{2}\times 100}{\left(1+\frac{y}{2}\right)^n} + T \times \frac{100}{\left(1+\frac{y}{2}\right)^{2T}}\right], \tag{2}$$

which is the negative of dollar change in bond price per unit change in yield. Given that a typical change in yield is measured in basis points, the often used DV01 measure scales the dollar measure by 10,000:

$$\text{DV01} = \text{Dollar Duration}/10,000\,,$$

which measures the negative change in bond price per one basis point change in yield.

Figure 7 plots the bond price P as a function of yield y for a ten-year bond with coupon rate of 6%. Effectively, it plots the relation between P and y in Equation (1). As we can see, P is inversely related to y: decreasing y is coupled with increasing P. Also, the relation is not linear. But if we would like to approximate the relation linearly, we can pick a level of y, say $y = 6\%$ and $P = \$100$ and draw a tangent line at that point. As you've been taught many times in the past, the slope is $\partial P/\partial y$ as calculated in Equation (2). In other words, the dollar duration is the negative of the slope.

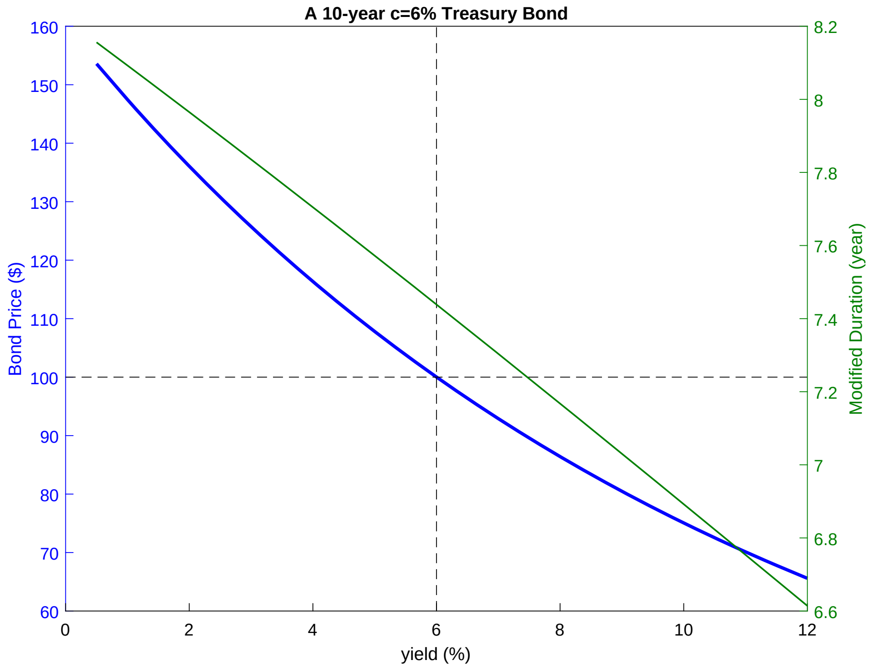

Figure 7: Bond Price as a Function of Yield and Duration as a Function of Yield

So if I would like to know how much I will lose when the ten-year Treasury yield suddenly increases by 10 basis points, I can use the linear approximation:

$$\Delta P_t = P_t - P_{t-1} \approx -\mathrm{D}^{\$} \times (y_t - y_{t-1}) = -\mathrm{D}^{\$} \times \Delta y_t = -D^{\$} \times \frac{10}{10,000} = -\mathrm{DV01} \times 10\,\mathrm{bps}$$

Going back to Figure 7, let's still focus just on the blue line. We notice that when y decreases, the slope gets steeper; when y increases, the slope gets flatter. This is because the relation between P and y as defined by Equation (1) is convex. For an investor holding a long position in bond, he would very much welcome this feature: profits due to decreasing y are amplified and losses due to increasing y are dampened.

- **Modified Duration:** The modified duration is defined as

$$-\frac{1}{P}\frac{\partial P}{\partial y} = \frac{1}{1+\frac{y}{2}} \frac{\sum_{n=1}^{2T} \frac{n}{2} \times \frac{\frac{c}{2}\times 100}{\left(1+\frac{y}{2}\right)^n} + T \times \frac{100}{\left(1+\frac{y}{2}\right)^{2T}}}{\sum_{n=1}^{2T} \frac{\frac{c}{2}\times 100}{\left(1+\frac{y}{2}\right)^n} + \frac{100}{\left(1+\frac{y}{2}\right)^{2T}}} \tag{3}$$

It is the dollar duration divided by the bond price. So its focus is on the profit/loss as a fraction of the position:

$$R_t = \frac{\Delta P_t}{P_{t-1}} = \frac{P_t - P_{t-1}}{P_{t-1}} - \approx \mathrm{D}^{\mathrm{mod}} \times (y_t - y_{t-1}) = -\mathrm{D}^{\mathrm{mod}} \times \Delta y_t$$

Dollar durations and modified durations are used for different purposes. If we are interested in the profit/loss in dollar terms, we go with the dollar duration, but if we interested in the profit/loss in the return space, we go with the modified duration.

As shown in Equation (3), the modified duration is a normalized measure and the unit is in year. In dealing with coupon bonds, it is always useful to go to the extreme and think first in terms of zero-coupon bond. For a T-year zero-coupon bond, the modified duration is T divided by $(1 + y/2)$. If instead of semi-annual compounding, the yield y is continuously compounded, then the modified duration of a T-year zero-coupon bond is simply T.

For a bond with semi-annual coupon payments, the modified duration is a weighted sum of all of the coupon payment dates, 0.5, 1.0, 1.5, ..., and T years. Except for the final date T, the n-th coupon dates are weighted by $\frac{c/2 \times 100}{(1+y/2)^n}$. The last date T carries a disproportionately high weight because of the principal payment \$100. Because of this, the weighting is always tilted toward the final date T. To be more precise, date T is weighted by $\frac{c/2 \times 100 + 100}{(1+y/2)^{2T}}$. For a coupon rate of 6%, $c/2 \times 100 + 100$ is 103, easily overpowering $c/2 = 3$.

You might wonder what happens when we have a really aggressive discount rate y, say $y = 10\%$? Well, let's consider the two extreme points: $\frac{1}{(1+y/2)^n}$ for the first coupon payment $n = 1$ and $\frac{1}{(1+y/2)^{2T}}$ for the final date T. Plugging $y = 10\%$, we have $\frac{1}{(1+y/2)} =$ 0.9524 and $\frac{1}{(1+y/2)^{2T}} = 0.3769$ for $T = 10$. As you can see, even with this very aggressive discount rate discounting over a 10-year period, the principal payment of \$100 still dominates the calculation.

This is why, as you can see in Table 3, the modified duration of a ten-year bond is close to 10, especially when y is low. As y gets higher, this discounting effect becomes relatively more important, pushing the "center of gravity" away from T. As a result, the modified duration gets smaller.

Building on this analogy of "center of gravity" a little bit more, let's go back to the picture in Figure 3, which is a useful picture to have in our head when doing bond math. At least this is how I do the math. I imagine that there is a center of gravity along the horizontal dimension. Its gets pulled/pushed left and right, depending on the

Table 3: Modified Duration

yield y	2%	3%	4%	5%	6%	6%	6%	7%	10%
coupon c	2%	3%	4%	5%	4.8%	6%	7.2%	7%	10%
$T = 1$	0.99	0.98	0.97	0.96	0.96	0.96	0.95	0.95	0.93
$T = 2$	1.95	1.93	1.90	1.88	1.87	1.86	1.84	1.84	1.77
$T = 3$	2.90	2.85	2.80	2.75	2.74	2.71	2.68	2.66	2.54
$T = 5$	4.74	4.61	4.49	4.38	4.36	4.27	4.18	4.16	3.86
$T = 7$	6.50	6.27	6.05	5.85	5.81	5.65	5.51	5.46	4.95
$T = 10$	9.02	8.58	8.18	7.79	7.71	7.44	7.21	7.11	6.23
$T = 20$	16.42	14.96	13.68	12.55	12.12	11.56	11.13	10.68	8.58
$T = 30$	22.48	19.69	17.38	15.45	14.46	13.84	13.39	12.47	9.46

relative weights between the last date T and the other coupon dates. Getting pushed to the left results in a smaller duration and getting pull to the right results in a larger duration.

For example, consider two bonds with the same y and same T but different coupon rate c. It could be that one bond was issued back in 1990 as a 30-year bond and has five year to maturity. The other bond is a newly issued 5-year notes. Assuming a flat term structure of interest rate, the yields of these two bonds are the same, but their coupon rates are different (so are their bond prices). Which one has a higher duration? The one with lower c has its center of gravity closer to T. As a result, it has a higher duration.

Generally, it is useful to have a table like that in Table 3 handy, or build a function in Excel to calculate the modified duration of a bond for give coupon c, yield y, and maturity T. Historically, the average 10-year yield is about 6%. It is useful to know that, for a 10-year par coupon bond with $c = 6\%$, its modified duration is around 7.44 years. (Not precisely 7.44, but a number around 7 or 8.) In recent years, interest rates have been low, implying a relatively high duration for bonds. Right now (Fall 2015), the 10-year yield is at 2.34%. It would be useful to know that a 10-year par coupon bond with $c = 2\%$ has a modified duration around 9 years. The current 5-year yield is at 1.72%, and it is useful to know that a 5-year par coupon bond with $c = 2\%$ has a modified duration around 4.75 years. There is no need to memorize these numbers, but to have a rough sense in terms of orders of magnitudes would be handy.

For example, we know that a typical one-day one-sigma move in 10-year yield is about 6.8 basis points. How much does that translate to return volatility? Recall that,

$R_t \approx \mathrm{D}^{\mathrm{mod}} \times \Delta y_t$. So, $\mathrm{std}(R_t) \approx \mathrm{D}^{\mathrm{mod}} \times \mathrm{std}\,(\Delta y_t)$. For a 10-year bond with a duration of 7.44, a 6.8-bps volatility in Δy_t translates to $6.8 \times 7.44 = 50.6$ basis points in daily return volatility. Right now (Fall 2015), in a low interest rate environment, duration is high. For the same amount of volatility in Δy_t, the bond return volatility would be higher because of the higher duration.

As another example, suppose you believe that the 30-year bond is priced cheap relative to the yield curve. Your model tells you that the spread between the 30-year bond and the curve (generated by your model) is about 10 basis points. You believe that this spread is due to temporary illiquidity in 30-year bonds and will converge to close to zero later on. How much does this 10 basis points translate to return? Right now (Fall 2015), the 30-year yield is at 3.12%. Table 3 tells us that at this rate, the modified duration is about 20 years. So $R_t \approx -\mathrm{D}^{\mathrm{mod}} \times \Delta y_t = -20 \times (-10\,\mathrm{bps}) = 2\%$.

- **Duration and Convexity:** Concepts such as duration and convexity are only meaning because we work in the yield space and the profit/loss is in the dollar space. As such, duration serves as a bridge that connects the bond price to yield:

  – Dollar Duration:

$$\Delta P_t = P_t - P_{t-1} \approx -\mathrm{D}^{\$} \times (y_t - y_{t-1}) = -\mathrm{D}^{\$} \times \Delta y_t$$

  – Modified Duration:

$$R_t = \frac{\Delta P_t}{P_{t-1}} = \frac{P_t - P_{t-1}}{P_{t-1}} - \approx \mathrm{D}^{\mathrm{mod}} \times (y_t - y_{t-1}) = -\mathrm{D}^{\mathrm{mod}} \times \Delta y_t$$

In addition to this linear approximation through duration, we also notice that the relation between price and yield is not linear but convex. So convexity is introduced as a second-order approximation to improve upon the first order, linear approximation. In this class, we will not go for the exact formula for this second order approximation. If one day, you become a bond trader/portfolio manager, than you might be busy with convexity hedging. Even then, you might notice that the term structure of interest rate is not flat, which could cause quite a bit of problem for your first order duration hedge.

Let me close by talking about one intuition associated with convexity that is important. The relation between duration and yield is as plotted in Figure 7. With decreasing y, duration increases. As a result, the profit from holding a bond gets amplified. This effect is not symmetric in losses because with increasing y, duration decreases. As a result, the loss associated with holding a bond gets dampened. This positive convexity makes bond more attractive than a security that is linear in y. Later on, we will see a fixed-income security (Mortgage-Backed Securities) with negative convexity and use bonds (with positive convexity) to do duration hedge.

3 The Universe of Fixed Income Securities

Fixed-income securities share one thing: exposures to the Treasury yield curve. Most of these securities have an added component of credit risk. Muni's are bonds issued by municipalities, whose default probability is higher than the US government. The recent bankruptcy of Detroit is one example. Corporate bonds are issued by individual corporations, which also include credit risk. Agency bonds are issued by the government sponsored agencies (GSE) like Fannie and Freddie. After the government takeover in 2008, these bonds are explicitly backed by the US government. Prior to the takeover, it was implicitly backed by the government. For most of the fixed-income securities, the Treasury yield curve serves as a benchmark. Credit-sensitive instruments such as corporate bonds are usually quoted in terms of its spread relative to the US treasury yield.

Table 4 gives a summary of the US bond market. It gives us a sense of the relative size of the various components of the fixed-income market. In later classes, we will study the corporate bond market and will also touch upon the mortgage backed securities.

Table 4: Outstanding US Bond Market Debt in $ Billions

	Muni	Treasury	Mortgage Related	Corp Debt	Agency Bonds	Money Markets	Asset Backed	Total
1980	399.4	623.2	111.4	458.6	164.3	480.7		2,237.7
1981	443.7	720.3	127.0	489.2	194.5	593.7		2,568.4
1982	508.0	881.5	177.1	534.7	208.8	622.7		2,932.8
1983	575.1	1,050.9	248.3	575.3	209.3	638.3		3,297.2
1984	650.6	1,247.4	302.9	651.9	240.4	777.1		3,870.4
1985	859.5	1,437.7	399.9	776.6	261.0	950.9	1.2	4,686.7
1986	920.4	1,619.0	614.7	959.3	276.6	998.6	11.3	5,399.8
1987	1,012.0	1,724.7	816.0	1,074.9	308.3	1,125.8	18.1	6,079.7
1988	1,080.0	1,821.3	973.6	1,195.8	370.7	1,263.0	25.8	6,730.1
1989	1,129.8	1,945.4	1,192.7	1,292.4	397.5	1,359.5	37.3	7,354.6
1990	1,178.6	2,195.8	1,340.1	1,350.3	421.5	1,328.9	66.2	7,881.5
1991	1,272.1	2,471.6	1,577.1	1,454.6	421.5	1,215.7	91.7	8,504.3
1992	1,295.4	2,754.1	1,774.3	1,557.1	462.4	1,157.9	116.4	9,117.6
1993	1,361.7	2,989.5	2,209.0	1,782.8	550.8	1,143.6	132.5	10,170.0
1994	1,325.8	3,126.0	2,352.9	1,931.1	727.7	1,229.1	161.9	10,854.5
1995	1,268.2	3,307.2	2,432.1	2,087.5	924.0	1,367.6	214.9	11,601.4
1996	1,261.6	3,459.7	2,606.4	2,247.9	925.8	1,572.7	296.8	12,371.0
1997	1,318.5	3,456.8	2,871.8	2,457.5	1,021.8	1,871.1	392.5	13,390.0
1998	1,402.7	3,355.5	3,243.4	2,779.4	1,302.1	2,091.9	477.8	14,652.8
1999	1,457.1	3,266.0	3,832.2	3,120.0	1,620.0	2,452.7	583.5	16,331.5
2000	1,480.7	2,951.9	4,119.3	3,400.5	1,853.7	2,815.8	699.5	17,321.5
2001	1,603.4	2,967.5	4,711.0	3,824.6	2,157.4	2,715.0	811.9	18,790.8
2002	1,762.8	3,204.9	5,286.3	4,035.5	2,377.7	2,637.2	902.0	20,206.3
2003	1,900.4	3,574.9	5,708.0	4,310.4	2,626.2	2,616.1	992.7	21,728.6
2004	2,821.2	3,943.6	6,289.1	4,537.9	2,700.6	2,996.1	1,096.6	24,385.1
2005	3,019.3	4,165.9	7,206.4	4,604.0	2,616.0	3,536.6	1,275.0	26,423.2
2006	3,189.3	4,322.9	8,376.0	4,842.5	2,634.0	4,140.0	1,642.7	29,147.3
2007	3,424.8	4,516.7	9,372.6	5,254.3	2,906.2	4,310.8	1,938.8	31,724.2
2008	3,517.2	5,783.6	9,457.6	5,417.5	3,210.6	3,939.3	1,799.3	33,125.2
2009	3,672.5	7,260.6	9,341.6	5,934.5	2,727.5	3,243.9	1,682.1	33,862.7
2010	3,772.1	8,853.0	9,221.4	6,543.4	2,538.8	2,980.8	1,476.3	35,385.9
2011	3,719.4	9,928.4	9,043.8	6,618.1	2,326.9	2,719.3	1,330.0	35,685.9
2012	3,714.4	11,046.1	8,814.9	7,049.6	2,095.8	2,612.3	1,253.6	36,586.7
2013	3,671.2	11,854.4	8,720.1	7,458.6	2,056.9	2,713.7	1,252.5	37,727.3
2014	3,652.4	12,504.8	8,729.4	7,846.2	2,028.7	2,903.3	1,336.5	39,001.3
2015Q1	3,694.0	12,630.2	8,688.9	7,965.1	1,975.6	2,879.2	1,361.3	39,194.4