Class 3: Options and Stock Market Crashes

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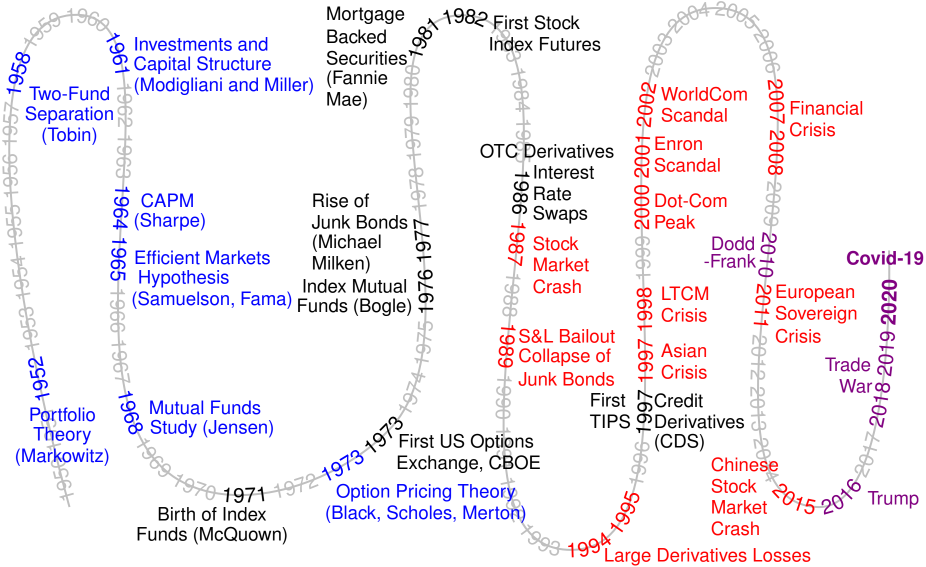
Outline

- Why Options?
  - The beginning of financial innovation.
  - New dimension of risk taking: the flexibility to take only the desired risk.
  - Market prices of such "carved out" risk contain unique information (e.g., VIX).
- The Black-Scholes option pricing model:
  - Pathbreaking framework: continuous-time arbitrage pricing.
  - Black-Scholes option implied volatility.
- Options and market crashes:
  - Out-of-money put options: highly sensitive to the left tail (i.e., crashes).
  - Their market prices: crash probability and fear of crash.
  - A model with market crash.

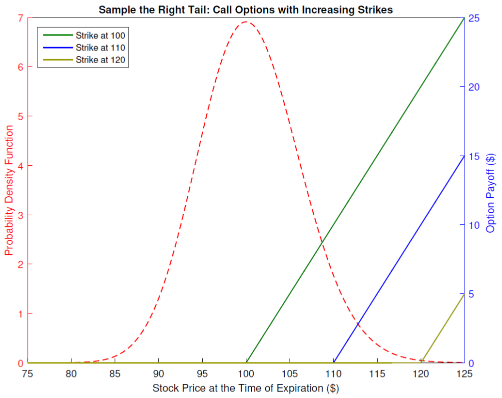
Modern Finance

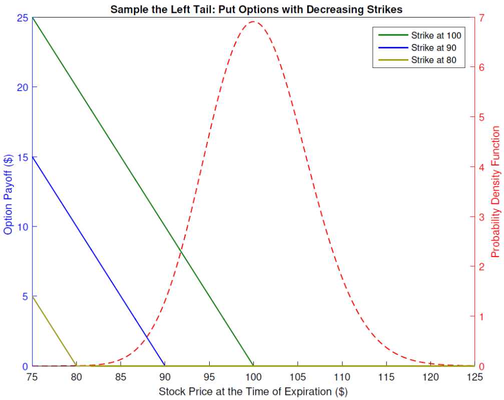


A Brief History

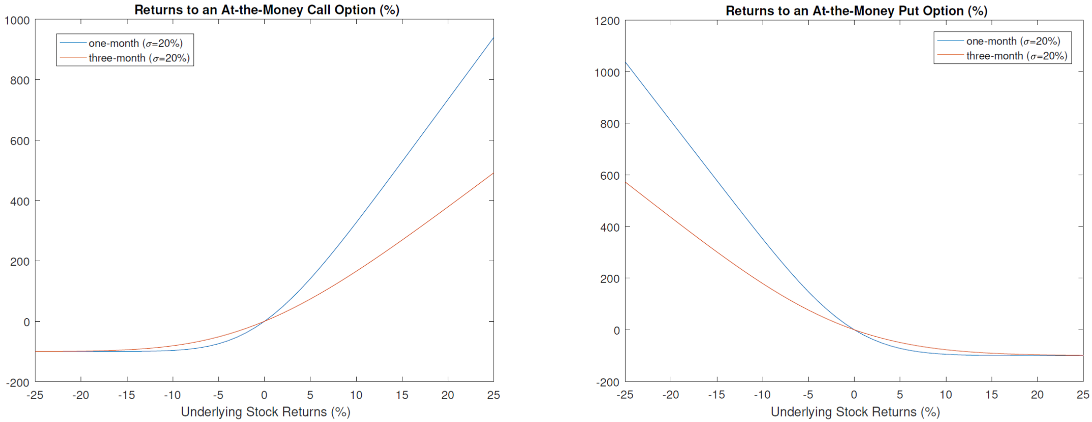
- 1973: CBOE founded as the first US options exchange, and 911 contracts were traded on 16 underlying stocks on first day of trading.
- 1975: The Black-Scholes model was adopted for pricing options.
- 1977: Trading in put options begins.
- 1983: On March 11, index option (OEX) trading begins; On July 1, options trading on the S&P 500 index (SPX) was launched.
- 1987: Stock market crash.
- 1993: Introduces CBOE Volatility Index (VIX).
- 2003: ISE (an options exchange founded in 2000) overtook CBOE to become the largest US equity options exchange.
- 2004: CBOE Launches futures on VIX.

Sampling the Tails





Leverage Embedded in Options



A Nobel-Prize Winning Formula

The Bank of Sweden Prize in Economic Sciences in Memory of Alfred Nobel 1997

"for a new method to determine the value of derivatives"

Robert C. Merton
1/2 of the prize
USA

Harvard University
Cambridge, MA, USA

b. 1944

Myron S. Scholes
1/2 of the prize
USA

Long Term Capital Management
Greenwich, CT, USA

b. 1941
(in Timmins, ON, Canada)

The Black-Scholes Model

- **The Model:** Let S_t be the time-t stock price, ex dividend. Prof. Black, Merton, and Scholes use a geometric Brownian motion to model S_t:

$$dS_t = (\mu - q)\, S_t\, dt + \sigma\, S_t\, dB_t\,.$$

- **Drift:** $(\mu - q)\, S_t\, dt$ is the deterministic component of the stock price. The stock price, ex dividend, grows at the rate of $\mu - q$ per year:
  - μ: expected stock return (continuously compounded), around 12% per year for the S&P 500 index.
  - q: dividend yield, round 2% per year for the S&P 500 index.
- **Diffusion:** $\sigma\, S_t\, dB_t$ is the random component, with B_t as a Brownian motion. σ is the stock return volatility, around 20% per year for the S&P 500 index.

Brownian Motion

- **Independence of increments:** For all $0 = t_0 < t_1 < \ldots < t_m$, the increments are independent:

$$B(t_1) - B(t_0),\ B(t_2) - B(t_1),\ \ldots,\ B(t_m) - B(t_{m-1})$$

  Translating to Finance: stock returns are independently distributed. No predictability and zero auto-correlation $\rho = 0$.

- **Stationary normal increments:** $B_t - B_s$ is normally distributed with zero mean and variance $t - s$.
  Translating to Finance: stock returns are normally distributed. Over a fixed horizon of T, return volatility is scaled by $\sqrt{T}$.

- **Continuity of paths:** $B(t)$, $t \geq 0$ are continuous functions of t.
  Translating to Finance: stock prices move in a continuous fashion. There are no jumps or discontinuities.

The Model in R_T

- It is more convenient to work in the log-return space:

$$R_T = \ln S_T - \ln S_0 \,, \text{ or equivalently, } S_T = S_0 e^{R_T}$$

- Using the model for S_T, we get

$$R_T = \left(\mu - q - \frac{1}{2}\sigma^2\right) T + \sigma\sqrt{T}\,\epsilon_T \,,$$

- Most of the terms are familiar to us:
  - $(\mu - q)T$ is the expected growth rate, ex dividend, over time T.
  - $\sigma\sqrt{T}$ is the stock return volatility over time T.
  - ϵ_T is a standard normal (inherited from the Brownian motion).
- The extra term of $-\frac{1}{2}\sigma^2 T$ is called the Ito's term. It needs to be there because the transformation from S_T to R_T involves taking a log, which is a non-linear (concave) function, of the random variable S_T.

Pricing a Call Option

- Option payoff $(S_T - K)^+$:
  - $S_T - K$ if $S_T > K$.
  - and zero otherwise.
- Option value = PV(payoff):

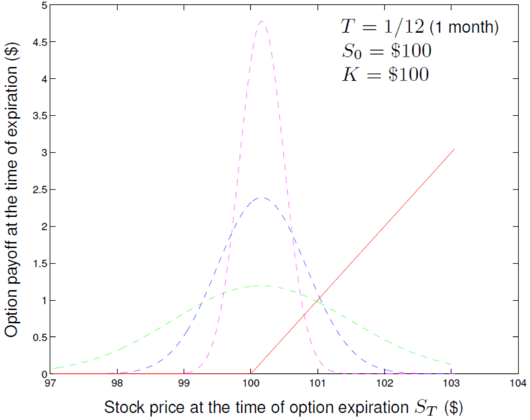
$$C_0 = E^Q\left(e^{-rT}(S_T - K)\mathbf{1}_{S_T>K}\right) ,$$

under risk-neutral measure Q.

- The Black-Scholes formula:

$$C_0 = e^{-qT}\,S_0\,N(d_1) - e^{-rT}\,K\,N(d_2) .$$

- At-the-money option: $\frac{C_0}{S_0} \approx \frac{1}{\sqrt{2\pi}}\,\sigma\sqrt{T}$.



The Insight of Arbitrage Pricing

- The key insight of arbitrage pricing is very simple: **replication.**
- A security offers me a stream of random payoffs:
  - If I can replicate that cash flow (no matter how random they might be), then the price tag equates the cost of replication.
  - Simple? In reality, it is difficult to find such exact replications.
  - This makes sense: Why do we need a security that can be replicated?
- An option offers a random payoff at the time of expiration T:
  - The most important insight: dynamic replication.
  - The limitation: the replication is done under the Black-Scholes model.
  - The pricing formula is valid if the assumptions of the model are true.

Risk-Neutral Pricing

- Risk-neutral pricing is a widely adopted tool in arbitrage pricing.
- Our model in the return space:

$$\text{P-measure:} \quad R_T = \left(\mu - q - \frac{1}{2}\sigma^2\right) T + \sigma\sqrt{T}\,\epsilon_T\,.$$

- In risk-neutral pricing, we bend the reality by making the stock grow instead at the riskfree rate r:

$$\text{Q-measure:} \quad R_T = \left(r - q - \frac{1}{2}\sigma^2\right) T + \sigma\sqrt{T}\,\epsilon_T^{\mathsf{Q}}$$

- Risk-neutral pricing: cash flows are discounted by the riskfree rate r and expectations are done under the Q-measure:

$$C_0 = E^{Q}\left(e^{-rT}(S_T - K)\mathbf{1}_{S_T > K}\right)$$

Pricing a Stock

- Consider the S&P 500 index and assume zero dividend $q = 0$. The index's final payoff is S_T. How much are you willing to pay for it today? Of course, S_0.
- Under P-measure:
$$e^{-\mu T} E^{\mathsf{P}}(S_T) = e^{-\mu T} S_0 e^{\mu T} = S_0$$
- Under Q-measure:
$$e^{-rT} E^{\mathsf{Q}}(S_T) = e^{-rT} S_0 e^{rT} = S_0$$
- Pricing using a Risk-neutral investor:
$$e^{-rT} E^{\mathsf{P}}(S_T) = e^{-rT} S_0 e^{\mu T} = S_0 \, e^{(\mu - r)T}$$
- Risk-neutral pricing does not mean pricing using a risk-neutral investor.

Pricing a Call Option

- Let C_0 be the present value of a European-style call option on S_T with strike price K. Using risk-neutral pricing:

$$C_0 = E^Q\left(e^{-rT}(S_T - K)\mathbf{1}_{S_T>K}\right)$$
$$= \boxed{e^{-rT}\,E^Q\left(S_T\mathbf{1}_{S_T>K}\right)} - \boxed{e^{-rT}\,K\,E^Q\left(\mathbf{1}_{S_T>K}\right)}$$

- Let's go directly to the solution (again assume $q = 0$ for simplicity):

$$C_0 = \boxed{S_0\,N(d_1)} - \boxed{e^{-rT}K\,N(d_2)},$$

where $N(d)$ is the cumulative distribution function of a standard normal.

- Comparing the terms in blue, we have $N\left(d_2\right) = E^Q(\mathbf{1}_{S_T>K})$, which is Prob$^Q\left(S_T > K\right)$, the probability that the option expires in the money under the Q-measure.
- Comparing the terms in green: $N\left(d_1\right) = e^{-rT}E^Q\left(\frac{S_T}{S_0}\,\mathbf{1}_{S_T>K}\right)$.

Understanding d_2 and d_1:

$$d_1 = \frac{\ln\left(S_0/K\right) + \left(r + \sigma^2/2\right)T}{\sigma\sqrt{T}}\,; \quad d_2 = \frac{\ln\left(S_0/K\right) + \left(r - \sigma^2/2\right)T}{\sigma\sqrt{T}}$$

- The model for S_T under Q-measure is $S_T = S_0\, e^{R_T}$ with

$$\text{Q-measure:} \quad R_T = \left(r - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}\,\epsilon_T^{\mathsf{Q}}$$

- We can verify that $N(d_2)$ indeed gives us Prob$^{Q}\left(S_T > K\right)$: the probability that the option expires in the money under the Q-measure.
- What about $N(d_1)$? With $E\left(S_T \mathbf{1}_{S_T>K}\right)$, it calculates the expectation of S_T only when $S_T > K$. This calculation is not required for exams.
- If you like, you can think of $N(d_1)$ as Prob$^{QQ}\left(S_T > K\right)$,

$$\text{QQ-measure:} \quad R_T = \left(r + \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}\,\epsilon_T^{\mathsf{QQ}}$$

The Black-Scholes Formula

- The Black-Scholes formula for a call option (bring dividend back),

$$C_0 = e^{-qT}\, S_0\, N(d_1) - e^{-rT}\, K\, N(d_2)$$

$$d_1 = \frac{\ln(S_0/K) + (r - q + \sigma^2/2)\, T}{\sigma\sqrt{T}}\,, \quad d_2 = \frac{\ln(S_0/K) + (r - q - \sigma^2/2)\, T}{\sigma\sqrt{T}}$$

- Put/call parity is model free. Holds even if the Black-Scholes model fails,

$$C_0 - P_0 = e^{-q\,T} S_0 - e^{-r\,T} K\,.$$

  Empirically, this relation holds well in the data and is similar in spirit to the arbitrage activity between the futures and cash markets.
- Using put/call parity, the Black-Scholes pricing formula for a put option is:

$$\begin{aligned} P_0 &= -e^{-qT}\, S_0\, (1 - N(d_1)) + e^{-r\,T}\, K\, (1 - N(d_2)) \\ &= -e^{-qT}\, S_0\, N(-d_1) + e^{-r\,T}\, K\, N(-d_2) \end{aligned}$$

At-the-Money Options

- For an at-the-money option, whose strike price is $K = S_0\, e^{(r-q)\,T}$

$$C_0 = P_0 = S_0 \left[N\left(\frac{1}{2}\sigma\sqrt{T}\right) - N\left(-\frac{1}{2}\sigma\sqrt{T}\right)\right]$$

- Recall that $N(d)$ is the cdf of a standard normal,

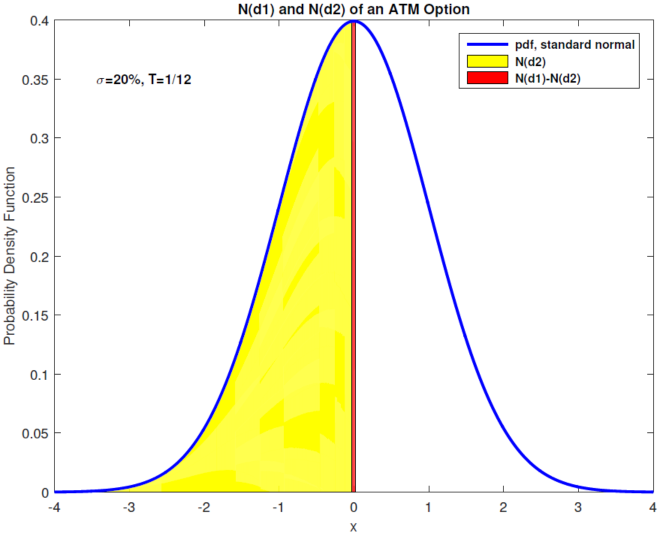
$$N(d) = \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx$$

- So the pricing formula can be further simplified to

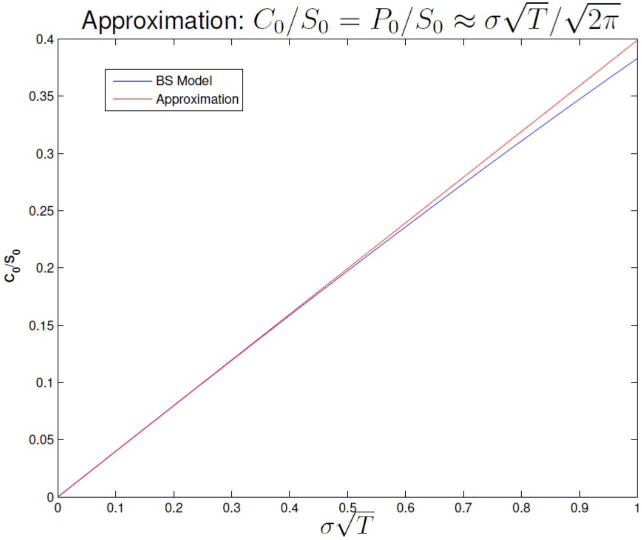
$$\frac{C_0}{S_0} = \frac{P_0}{S_0} = \int_{-\frac{1}{2}\sigma\sqrt{T}}^{\frac{1}{2}\sigma\sqrt{T}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx \approx \frac{1}{\sqrt{2\pi}}\, \sigma\sqrt{T}\,,$$

which works well for small $\sigma\sqrt{T}$. For large $\sigma\sqrt{T}$ (volatile markets or long-dated options), non-linearity becomes important and this approximation is imprecise.

ATM Options: $d_1 = \frac{1}{2}\sigma\sqrt{T}$ and $d_2 = -\frac{1}{2}\sigma\sqrt{T}$



ATM Options as a Linear Contract on $\sigma\sqrt{T}$



The Black-Scholes Option Implied Volatility

- At time 0, a call option struck at K and expiring on date T is traded at C_0. At the same time, the underlying stock price is traded at S_0, and the riskfree rate is r.
- If we know the market volatility σ at time 0, we can apply the Black-Scholes formula:

$$C_0^{\text{Model}} = \text{BS}(S_0, K, T, \sigma, r, q)$$

- Volatility is something that we don't observe directly. But using the market-observed price C_0^{Market}, we can back it out:

$$C_0^{\text{Market}} = C_0^{\text{Model}} = \text{BS}(S_0, K, T, \sigma^I, r, q)\,.$$

- If the Black-Scholes model is the correct model, then the Option Implied Volatility σ^I should be exactly the same as the true volatility σ.

SPX Options with Varying Moneyness

On March 2, 2006, the following SPX put options are traded on CBOE:

P_0	S_0	K	OTM-ness	T	σ^I	P_0^{BS}
9.30	1287	1285	0.15%	16/365	10.06%	?
6.00	1287	1275	0.93%	16/365	10.64%	5.44
2.20	1287	1250	2.87%	16/365	12.74%	0.92
1.20	1287	1225	4.82%	16/365	15.91%	0.075
1.00	1287	1215	5.59%	16/365	17.24%	0.022
0.40	1287	1170	9.09%	16/365	22.19%	0.000013

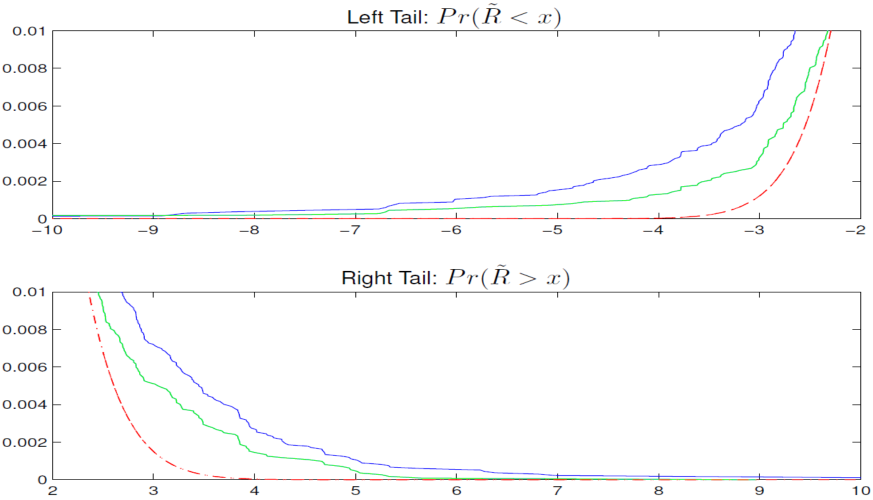
P_0^{BS} is the Black-Scholes price assuming $\sigma = 10.06\%$.

Expected Option Returns

Strike - Spot	-15 to -10	-10 to -5	-5 to 0	0 to 5	5 to 10
Weekly SPX Put Option Returns (in %)					
mean return	-14.56	-12.78	-9.50	-7.71	-6.16
max return	475.88	359.18	307.88	228.57	174.70
min return	-84.03	-84.72	-87.72	-88.90	-85.98
mean BS β	-36.85	-37.53	-35.23	-31.11	-26.53
corrected return	-10.31	-8.45	-5.44	-4.12	-3.10

Coval and Shumway, *Journal of Finance*, 2000. Data from Jan. 1990 through Oct. 1995.

Tail Distributions: Model vs Data



Crash and Crash Premium

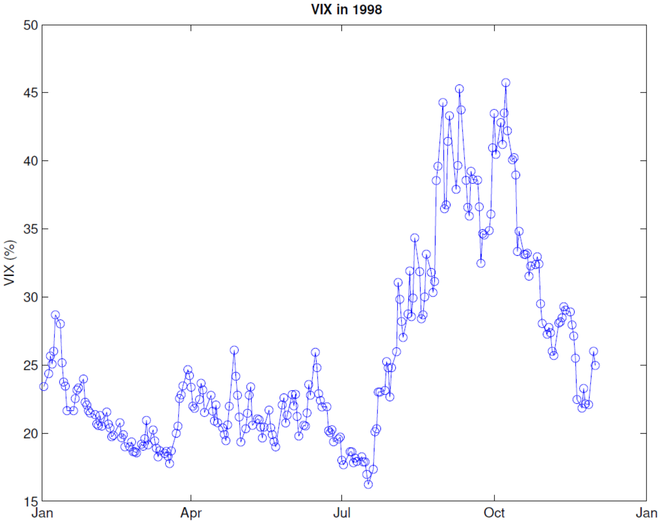
- Selling volatility and selling crash insurance are profitable, and their risk profile differs significantly from that of stock portfolios.
- In the presence of tail risk, options are no longer redundant and cannot be dynamically replicated, and their pricing has two components:
  - the likelihood and magnitude of the tail risk.
  - aversion or preference toward such tail events.
- The "over-pricing" of put options on the S&P 500 index reflects not only the probability and severity of market crashes, but also investors' aversion to such crashes — crash premium.
- In fact, the crash premium accounts for most of the "over-pricing" in short-dated OTM puts and ATM options.

The Bank of Volatility

Excerpts from *"When Genius Failed"* by Roger Lowenstein

- Early in 1998, LTCM began to short large amounts of equity volatility.
- Betting that implied volatility would eventually revert to its long-run mean of 15%, they shorted options at prices with an implied volatility of 19%.
- Their position is such that each percentage change in implied vol will make or lose $40 million in their option portfolio.
- Morgan Stanley coined a nickname for the fund: the Central Bank of Volatility.

VIX in 1998



Implications for the 2008 Crisis

- The OTM put options on the S&P 500 index is a very good example for us to remember what an insurance on the market looks like.
- So next time when you see one, you will recognize it for what it is.
- As we learned from the recent crisis, some supposedly sophisticated investors wrote insurance on the market without knowing, the willingness to know, or the integrity to acknowledge the consequences.
- $0 \times \$100$ billion $= 0$, but only if the zero is really zero.
- Small probability events have a close to zero probability, but not zero!
- So $10^{-9} \times \$100$ billion $\neq 0$! And the math is in fact more complicated.
- And if this small probability event has a market-wide impact, then you need to be very careful.

Excerpts from *Fool's Gold* by Gillian Tett

- By 2006, Merrill topped the league table in terms of underwriting CDO's, selling a total of $52 billion that year, up from $2 billion in 2001.
- Behind the scenes, Merrill was facing the same problem that worried Winters at J.P.Morgan: what to do with the super-senior debt?
- Initially, Merrill solved the problem by buying insurance for its super-senior debt from AIG.
- In late 2005, AIG told Merrill it would no longer offer that service.
- The CDO team decided to start keeping the risk on Merrill's books.
- In 2006, sales of the various CDO notes produced some $700 million worth of fees. Meanwhile, the retained super-senior rose by more than $5 billion each quarter.

Excerpts from *Fool's Gold* by Gillian Tett

- As the CDS team posted more and more profits, it became increasingly difficult for other departments, or even risk controllers, to interfere.
- O'Neal himself could have weighted in, but he was in no position to discuss the finer details of super-senior risk.
- The risk department did not even report directly to the board.
- O'Neal faces absolutely no regulatory pressure to manage the risk any better.
- Far from it. The main regulator of the brokerages was the SEC, which had recently removed some of the old constraints.

Excerpts from *Fool's Gold* by Gillian Tett

- Citigroup was also keen to ramp up the output of its CDO machine.
- Unlike the brokerages, though, Citi could not park unlimited quantities of super-senior on its balance sheet, since the US regulatory system did still impose a leverage limit on commercial banks.
- Citi decided to circumvent that rule by placing large volumes of its super-senior in an extensive network of SIVs and other off balance sheet vehicles that it created.
- The SIVs were not always eager to buy the risk, so Citi began throwing in a type of "buyback" sweetener: it promised that if the SIVs ever ran into problems with the super-senior notes, Citi itself would buy them back.
- By 2007, it had extended such "liquidity puts" on $25 billion of super-senior notes. It also held more than $10 billion of the notes on its own books.