Class 10: Options and Stock Market Crashes

Financial Markets, Spring 2020, SAIF

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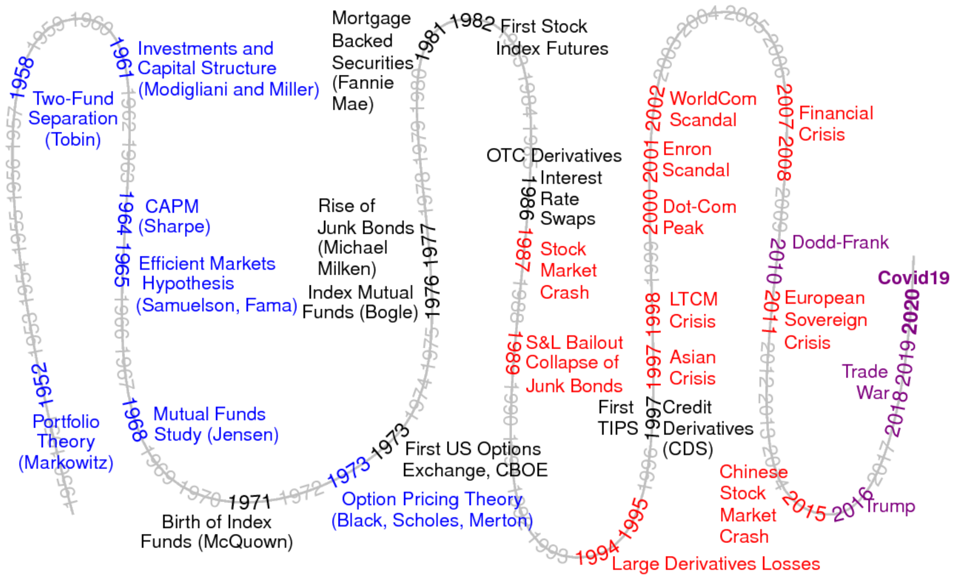
April 12, 2020

Outline

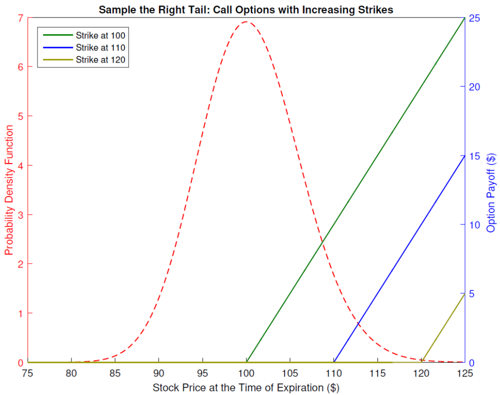
- Why Options?
  - The beginning of financial innovation.
  - New dimension of risk taking: the flexibility to take only the desired risk.
  - Market prices of such "carved out" risk contain unique information (e.g., VIX).
- The Black-Scholes option pricing model:
  - Pathbreaking framework: continuous-time arbitrage pricing.
  - Black-Scholes option implied volatility.
- Options and market crashes:
  - Out-of-money put options: highly sensitive to the left tail (i.e., crashes).
  - Their market prices: crash probability and fear of crash.
  - A model with market crash.

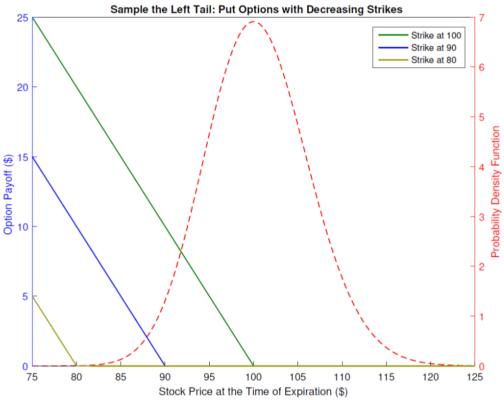

1958
Two-Fund Separation (Tobin)
1961
Investments and Capital Structure (Modigliani and Miller)
1952
Portfolio Theory (Markowitz)
1964
CAPM (Sharpe)
1965
Efficient Markets Hypothesis (Samuelson, Fama)
1968
Mutual Funds Study (Jensen)
1971
Birth of Index Funds (McQuown)
1973
Option Pricing Theory (Black, Scholes, Merton)
1973
First US Options Exchange, CBOE
1976
Index Mutual Funds (Bogle)
1977
Rise of Junk Bonds (Michael Milken)
1981
Mortgage Backed Securities (Fannie Mae)
1982
First Stock Index Futures
1986
OTC Derivatives
Interest Rate Swaps
1987
Stock Market Crash
1989
S&L Bailout
Collapse of Junk Bonds
1994
1995
Large Derivatives Losses
1997
First TIPS
Credit Derivatives (CDS)
1997
Asian Crisis
1998
LTCM Crisis
2000
Dot-Com Peak
2001
Enron Scandal
2002
WorldCom Scandal
2007
2008
Financial Crisis
2010
Dodd-Frank
2011
European Sovereign Crisis
2015
Chinese Stock Market Crash
2016
Trump
2018
2019
Trade War
2020
Covid19

A Brief History

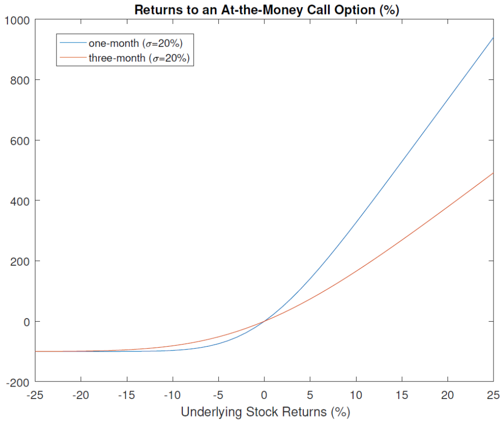
- 1973: CBOE founded as the first US options exchange, and 911 contracts were traded on 16 underlying stocks on first day of trading.
- 1975: The Black-Scholes model was adopted for pricing options.
- 1977: Trading in put options begins.
- 1983: On March 11, index option (OEX) trading begins; On July 1, options trading on the S&P 500 index (SPX) was launched.
- 1987: Stock market crash.
- 1993: Introduces CBOE Volatility Index (VIX).
- 2003: ISE (an options exchange founded in 2000) overtook CBOE to become the largest US equity options exchange.
- 2004: CBOE Launches futures on VIX.

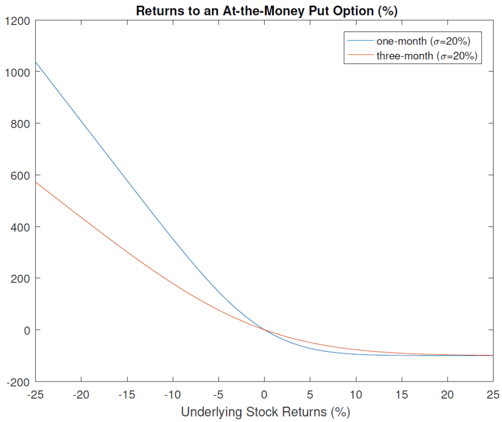
Sampling the Tails





Leverage Embedded in Options





A Nobel-Prize Winning Formula

The Bank of Sweden Prize in Economic Sciences in Memory of Alfred Nobel 1997

"for a new method to determine the value of derivatives"

Robert C. Merton

1/2 of the prize

USA

Harvard University
Cambridge, MA, USA

b. 1944

Myron S. Scholes

1/2 of the prize

USA

Long Term Capital Management
Greenwich, CT, USA

b. 1941
(in Timmins, ON, Canada)

The Black-Scholes Model

- **The Model:** Let S_t be the time-t stock price, ex dividend. Prof. Black, Merton, and Scholes use a geometric Brownian motion to model S_t:

$$dS_t = (\mu - q)\, S_t\, dt + \sigma\, S_t\, dB_t\,.$$

- **Drift:** $(\mu - q)\, S_t\, dt$ is the deterministic component of the stock price. The stock price, ex dividend, grows at the rate of $\mu - q$ per year:
  - μ: expected stock return (continuously compounded), around 12% per year for the S&P 500 index.
  - q: dividend yield, round 2% per year for the S&P 500 index.
- **Diffusion:** $\sigma\, S_t\, dB_t$ is the random component, with B_t as a Brownian motion. σ is the stock return volatility, around 20% per year for the S&P 500 index.

Brownian Motion

- **Independence of increments:** For all $0 = t_0 < t_1 < \ldots < t_m$, the increments are independent:

$$B(t_1) - B(t_0),\ B(t_2) - B(t_1),\ \ldots,\ B(t_m) - B(t_{m-1})$$

  Translating to Finance: stock returns are independently distributed. No predictability and zero auto-correlation $\rho = 0$.

- **Stationary normal increments:** $B_t - B_s$ is normally distributed with zero mean and variance $t - s$.
  Translating to Finance: stock returns are normally distributed. Over a fixed horizon of T, return volatility is scaled by $\sqrt{T}$.

- **Continuity of paths:** $B(t)$, $t \geq 0$ are continuous functions of t.
  Translating to Finance: stock prices move in a continuous fashion. There are no jumps or discontinuities.

The Model in R_T

- It is more convenient to work in the log-return space:

$$R_T = \ln S_T - \ln S_0\,, \text{ or equivalently, } S_T = S_0 e^{R_T}$$

- Using the model for S_T, we get

$$R_T = \left(\mu - q - \frac{1}{2}\sigma^2\right) T + \sigma\sqrt{T}\epsilon_T\,,$$

- Most of the terms are familiar to us:
  - $(\mu - q)T$ is the expected growth rate, ex dividend, over time T.
  - $\sigma\sqrt{T}$ is the stock return volatility over time T.
  - ϵ_T is a standard normal (inherited from the Brownian motion).
- The extra term of $-\frac{1}{2}\sigma^2 T$ is called the Ito's term. It needs to be there because the transformation from S_T to R_T involves taking a log, which is a non-linear (concave) function, of the random variable S_T.

Pricing a Call Option

- Option payoff $(S_T - K)^+$:
  - $S_T - K$ if $S_T > K$.
  - and zero otherwise.
- Option value = PV(payoff):

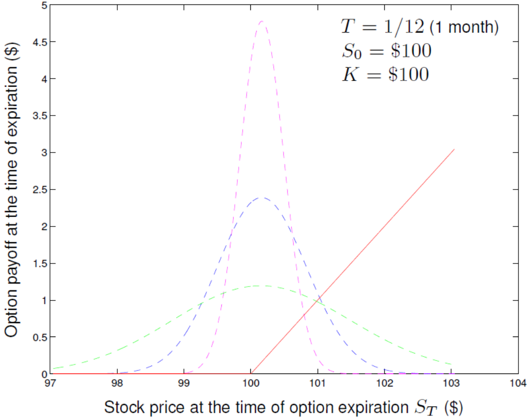
$$C_0 = E^Q\left(e^{-rT}(S_T - K)\mathbf{1}_{S_T>K}\right) ,$$

under risk-neutral measure Q.

- The Black-Scholes formula:

$$C_0 = e^{-qT} S_0 \, N(d_1) - e^{-rT} K \, N(d_2) \, .$$

- At-the-money option: $\frac{C_0}{S_0} \approx \frac{1}{\sqrt{2\pi}} \sigma\sqrt{T}$.



The Insight of Arbitrage Pricing

- The key insight of arbitrage pricing is very simple: **replication.**
- A security offers me a stream of random payoffs:
  - If I can replicate that cash flow (no matter how random they might be), then the price tag equates the cost of replication.
  - Simple? In reality, it is difficult to find such exact replications.
  - This makes sense: Why do we need a security that can be replicated?
- An option offers a random payoff at the time of expiration T:
  - The most important insight: dynamic replication.
  - The limitation: the replication is done under the Black-Scholes model.
  - The pricing formula is valid if the assumptions of the model are true.

Risk-Neutral Pricing

- Risk-neutral pricing is a widely adopted tool in arbitrage pricing.
- Our model in the return space:

$$\text{P-measure:} \quad R_T = \left(\mu - q - \frac{1}{2}\sigma^2\right) T + \sigma\sqrt{T}\epsilon_T \,.$$

- In risk-neutral pricing, we bend the reality by making the stock grow instead at the riskfree rate r:

$$\text{Q-measure:} \quad R_T = \left(r - q - \frac{1}{2}\sigma^2\right) T + \sigma\sqrt{T}\epsilon_T^{Q}$$

- Risk-neutral pricing: cash flows are discounted by the riskfree rate r and expectations are done under the Q-measure:

$$C_0 = E^{Q}\left(e^{-rT}(S_T - K)\mathbf{1}_{S_T > K}\right)$$

Pricing a Stock

- Consider the S&P 500 index and assume zero dividend $q = 0$. The index's final payoff is S_T. How much are you willing to pay for it today? Of course, S_0.
- Under P-measure:

$$e^{-\mu T} E^{P}(S_T) = e^{-\mu T} S_0 e^{\mu T} = S_0$$

- Under Q-measure:

$$e^{-rT} E^{Q}(S_T) = e^{-rT} S_0 e^{rT} = S_0$$

- Pricing using a Risk-neutral investor:

$$e^{-rT} E^{P}(S_T) = e^{-rT} S_0 e^{\mu T} = S_0 \, e^{(\mu - r)T}$$

- Risk-neutral pricing does not mean pricing using a risk-neutral investor.

Pricing a Call Option

- Let C_0 be the present value of a European-style call option on S_T with strike price K. Using risk-neutral pricing:

$$
\begin{aligned}
C_0 &= E^Q\left(e^{-rT}(S_T - K)\mathbf{1}_{S_T>K}\right) \\
&= \boxed{e^{-rT}\,E^Q\left(S_T\mathbf{1}_{S_T>K}\right)} - \boxed{e^{-rT}\,K\,E^Q\left(\mathbf{1}_{S_T>K}\right)}
\end{aligned}
$$

- Let's go directly to the solution (again assume $q = 0$ for simplicity):

$$
C_0 = \boxed{S_0\,N(d_1)} - \boxed{e^{-rT}KN(d_2)}\,,
$$

  where $N(d)$ is the cumulative distribution function of a standard normal.
- Comparing the terms in blue, we have $N(d_2) = E^Q(\mathbf{1}_{S_T>K})$, which is $\text{Prob}^Q(S_T > K)$, the probability that the option expires in the money under the Q-measure.
- Comparing the terms in green: $N(d_1) = e^{-rT}E^Q\left(\frac{S_T}{S_0}\mathbf{1}_{S_T>K}\right)$.

Understanding d_2 and d_1:

$$d_1 = \frac{\ln\left(S_0/K\right) + \left(r + \sigma^2/2\right)\, T}{\sigma\sqrt{T}}\,; \quad d_2 = \frac{\ln\left(S_0/K\right) + \left(r - \sigma^2/2\right)\, T}{\sigma\sqrt{T}}$$

- The model for S_T under Q-measure is $S_T = S_0\, e^{R_T}$ with

$$\text{Q-measure:} \quad R_T = \left(r - \frac{1}{2}\sigma^2\right) T + \sigma\sqrt{T}\,\epsilon_T^{\mathrm{Q}}$$

- We can verify that $N(d_2)$ indeed gives us $\text{Prob}^{Q}\left(S_T > K\right)$: the probability that the option expires in the money under the Q-measure.
- What about $N(d_1)$? With $E\left(S_T \mathbf{1}_{S_T>K}\right)$, it calculates the expectation of S_T only when $S_T > K$. This calculation is not required for exams.
- If you like, you can think of $N(d_1)$ as $\text{Prob}^{QQ}\left(S_T > K\right)$,

$$\text{QQ-measure:} \quad R_T = \left(r + \frac{1}{2}\sigma^2\right) T + \sigma\sqrt{T}\,\epsilon_T^{\mathrm{QQ}}$$

The Black-Scholes Formula

- The Black-Scholes formula for a call option (bring dividend back),

$$C_0 = e^{-qT}\, S_0\, N(d_1) - e^{-rT}\, K\, N(d_2)$$

$$d_1 = \frac{\ln\left(S_0/K\right) + \left(r - q + \sigma^2/2\right)\, T}{\sigma\sqrt{T}}, \quad d_2 = \frac{\ln\left(S_0/K\right) + \left(r - q - \sigma^2/2\right)\, T}{\sigma\sqrt{T}}$$

- Put/call parity is model free. Holds even if the Black-Scholes model fails,

$$C_0 - P_0 = e^{-q\,T} S_0 - e^{-r\,T} K\,.$$

  Empirically, this relation holds well in the data and is similar in spirit to the arbitrage activity between the futures and cash markets.

- Using put/call parity, the Black-Scholes pricing formula for a put option is:

$$\begin{aligned} P_0 &= -e^{-qT}\, S_0\, \left(1 - N(d_1)\right) + e^{-r\,T}\, K\, \left(1 - N(d_2)\right) \\ &= -e^{-qT}\, S_0\, N(-d_1) + e^{-r\,T}\, K\, N(-d_2) \end{aligned}$$

At-the-Money Options

- For an at-the-money option, whose strike price is $K = S_0\, e^{(r-q)\,T}$

$$C_0 = P_0 = S_0 \left[N\left(\frac{1}{2}\sigma\sqrt{T}\right) - N\left(-\frac{1}{2}\sigma\sqrt{T}\right)\right]$$

- Recall that $N(d)$ is the cdf of a standard normal,

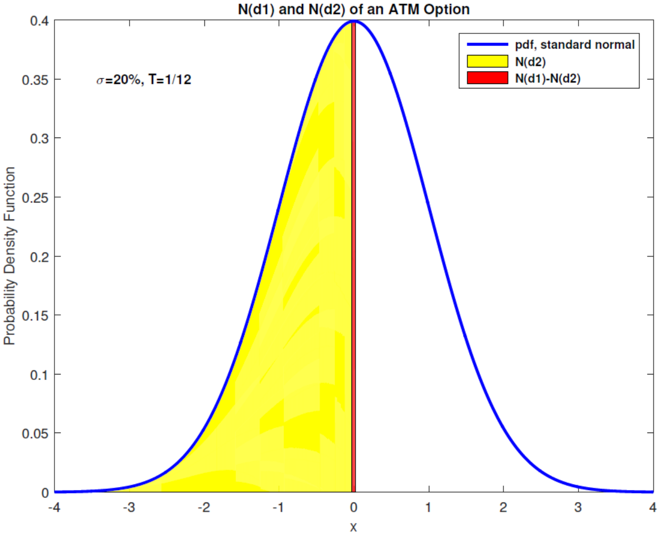
$$N(d) = \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx$$

- So the pricing formula can be further simplified to

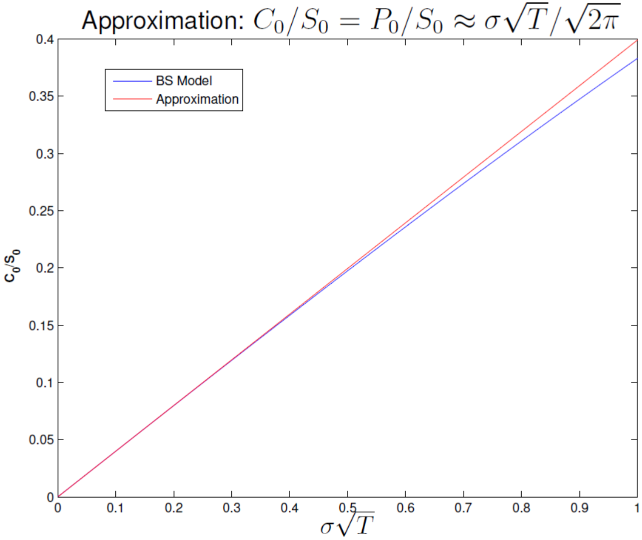
$$\frac{C_0}{S_0} = \frac{P_0}{S_0} = \int_{-\frac{1}{2}\sigma\sqrt{T}}^{\frac{1}{2}\sigma\sqrt{T}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx \approx \frac{1}{\sqrt{2\pi}}\, \sigma\sqrt{T},$$

which works well for small $\sigma\sqrt{T}$. For large $\sigma\sqrt{T}$ (volatile markets or long-dated options), non-linearity becomes important and this approximation is imprecise.

ATM Options: $d_1 = \frac{1}{2}\sigma\sqrt{T}$ and $d_2 = -\frac{1}{2}\sigma\sqrt{T}$



ATM Options as a Linear Contract on $\sigma\sqrt{T}$



Review: The Black-Scholes Option Pricing Model

The Black-Scholes Option Implied Volatility

- At time 0, a call option struck at K and expiring on date T is traded at C_0. At the same time, the underlying stock price is traded at S_0, and the riskfree rate is r.
- If we know the market volatility σ at time 0, we can apply the Black-Scholes formula:

$$\boxed{C_0^{\text{Model}} = \text{BS}(S_0, K, T, \sigma, r, q)}$$

- Volatility is something that we don't observe directly. But using the market-observed price C_0^{Market}, we can back it out:

$$C_0^{\text{Market}} = C_0^{\text{Model}} = \text{BS}(S_0, K, T, \sigma^I, r, q)\,.$$

- If the Black-Scholes model is the correct model, then the Option Implied Volatility σ^I should be exactly the same as the true volatility σ.

SPX Options with Varying Moneyness

On March 2, 2006, the following SPX put options are traded on CBOE:

P_0	S_0	K	OTM-ness	T	σ^I	P_0^{BS}
9.30	1287	1285	0.15%	16/365	10.06%	?
6.00	1287	1275	0.93%	16/365	10.64%	5.44
2.20	1287	1250	2.87%	16/365	12.74%	0.92
1.20	1287	1225	4.82%	16/365	15.91%	0.075
1.00	1287	1215	5.59%	16/365	17.24%	0.022
0.40	1287	1170	9.09%	16/365	22.19%	0.000013

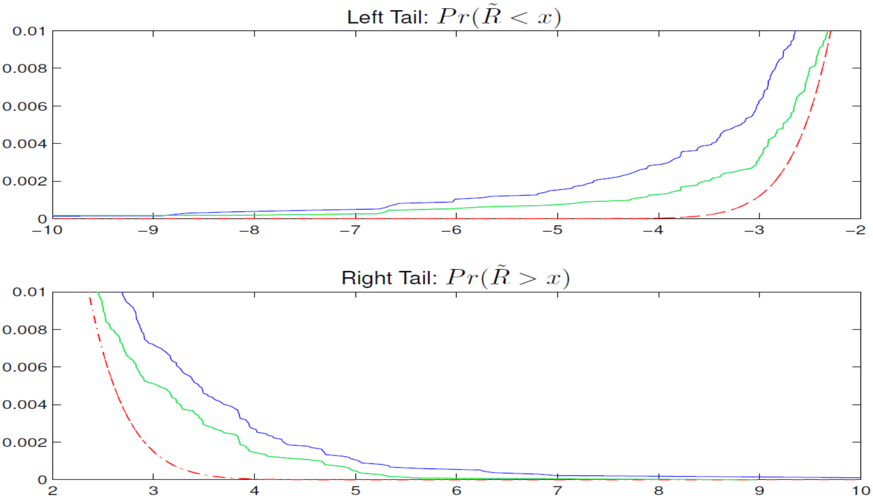
P_0^{BS} is the Black-Scholes price assuming $\sigma = 10.06\%$.

Expected Option Returns

Strike - Spot	-15 to -10	-10 to -5	-5 to 0	0 to 5	5 to 10
Weekly SPX Put Option Returns (in %)					
mean return	-14.56	-12.78	-9.50	-7.71	-6.16
max return	475.88	359.18	307.88	228.57	174.70
min return	-84.03	-84.72	-87.72	-88.90	-85.98
mean BS β	-36.85	-37.53	-35.23	-31.11	-26.53
corrected return	-10.31	-8.45	-5.44	-4.12	-3.10

Coval and Shumway, *Journal of Finance*, 2000. Data from Jan. 1990 through Oct. 1995.

Tail Distributions: Model vs Data



Crash and Crash Premium

- Selling volatility and selling crash insurance are profitable, and their risk profile differs significantly from that of stock portfolios.
- In the presence of tail risk, options are no longer redundant and cannot be dynamically replicated, and their pricing has two components:
  - the likelihood and magnitude of the tail risk.
  - aversion or preference toward such tail events.
- The "over-pricing" of put options on the S&P 500 index reflects not only the probability and severity of market crashes, but also investors' aversion to such crashes — crash premium.
- In fact, the crash premium accounts for most of the "over-pricing" in short-dated OTM puts and ATM options.

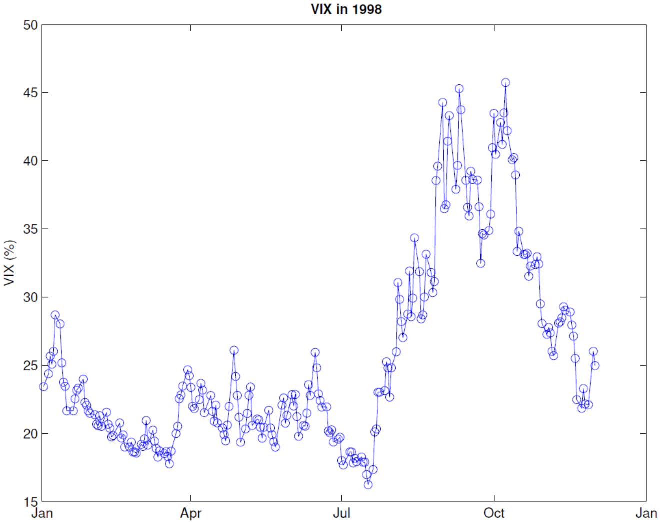
Review: Options and Market Crashes

The Bank of Volatility

Excerpts from *"When Genius Failed"* by Roger Lowenstein

- Early in 1998, LTCM began to short large amounts of equity volatility.
- Betting that implied volatility would eventually revert to its long-run mean of 15%, they shorted options at prices with an implied volatility of 19%.
- Their position is such that each percentage change in implied vol will make or lose $40 million in their option portfolio.
- Morgan Stanley coined a nickname for the fund: the Central Bank of Volatility.

VIX in 1998



Implications for the 2008 Crisis

- The OTM put options on the S&P 500 index is a very good example for us to remember what an insurance on the market looks like.
- So next time when you see one, you will recognize it for what it is.
- As we learned from the recent crisis, some supposedly sophisticated investors wrote insurance on the market without knowing, the willingness to know, or the integrity to acknowledge the consequences.
- $0 \times \$100\,\text{billion} = 0$, but only if the zero is really zero.
- Small probability events have a close to zero probability, but not zero!
- So $10^{-9} \times \$100\,\text{billion} \neq 0$! And the math is in fact more complicated.
- And if this small probability event has a market-wide impact, then you need to be very careful.

Excerpts from *Fool's Gold* by Gillian Tett

- By 2006, Merrill topped the league table in terms of underwriting CDO's, selling a total of $52 billion that year, up from $2 billion in 2001.
- Behind the scenes, Merrill was facing the same problem that worried Winters at J.P.Morgan: what to do with the super-senior debt?
- Initially, Merrill solved the problem by buying insurance for its super-senior debt from AIG.
- In late 2005, AIG told Merrill it would no longer offer that service.
- The CDO team decided to start keeping the risk on Merrill's books.
- In 2006, sales of the various CDO notes produced some $700 million worth of fees. Meanwhile, the retained super-senior rose by more than $5 billion each quarter.

Excerpts from *Fool's Gold* by Gillian Tett

- As the CDS team posted more and more profits, it became increasingly difficult for other departments, or even risk controllers, to interfere.
- O'Neal himself could have weighted in, but he was in no position to discuss the finer details of super-senior risk.
- The risk department did not even report directly to the board.
- O'Neal faces absolutely no regulatory pressure to manage the risk any better.
- Far from it. The main regulator of the brokerages was the SEC, which had recently removed some of the old constraints.

Excerpts from *Fool's Gold* by Gillian Tett

- Citigroup was also keen to ramp up the output of its CDO machine.
- Unlike the brokerages, though, Citi could not park unlimited quantities of super-senior on its balance sheet, since the US regulatory system did still impose a leverage limit on commercial banks.
- Citi decided to circumvent that rule by placing large volumes of its super-senior in an extensive network of SIVs and other off balance sheet vehicles that it created.
- The SIVs were not always eager to buy the risk, so Citi began throwing in a type of "buyback" sweetener: it promised that if the SIVs ever ran into problems with the super-senior notes, Citi itself would buy them back.
- By 2007, it had extended such "liquidity puts" on $25 billion of super-senior notes. It also held more than $10 billion of the notes on its own books.

A Model with Market Crash

- In Group Project 2, we work with a simplified version of Merton (1976). In that model, we have two additional parameters for the crash component: the one-month probability of "jump" ($p = 2\%$) and the "jump size" given its arrival (jump size = -20%).
- In Merton (1976), the jump arrival is dictated by a Poisson process with a jump arrival intensity of λ. Over a one-month horizon, the jump probability is $p = 1 - e^{-\lambda T}$, where $T = 1/12$. So $p = 2\%$ implies a jump intensity of $\lambda = 24.24\%$ per year.
- In Merton (1976), the jump size is normally distributed. So given jump arrival, there is uncertainty in jump amplitude. In our simplified model, we work with a constant jump size of -20%.
- In Merton (1976), the option pricing formula builds on the Black-Scholes model. For convenience, we use the cumbersome method of simulation.

What We Learned from the Crash Model?

- We find that in order to generate realistic volatility smirk to match the options data, we need the market to crash much more often than what has been historically observed.
- Conversely, if we plug into the model more realistic jump parameters (moderate p and jump size), then the model cannot generate the steep option-implied smirk as observed in the options data.
- In other words, investors are willing to pay a very high premium to have the crash risk hedged out of their portfolio. Conversely, selling OTM put options on the market can be a "good" investment strategy if you believe that such people suffers from "paranoia."
- Then rare events such as 2008 happens, and you realize that such "paranoia" is in fact rational: the "over-pricing" or the extra premium is due to a high level of risk aversion towards market crashes.

Main Takeaways