Class 12: Bond Math, Yield and Duration

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Outline

- Maturity and Coupon:
  - A bond matures. At maturity, the bond pays back the principal.
  - Before maturity, it has scheduled coupon payments.
- Duration:
  - Duration: measures the interest-rate exposure of a bond.
  - Very often, we will refer to buying bonds as buying duration.
  - So **beta** in equity, and **duration** in fixed income.
- Yield:
  - A more convenient measure of bond prices.
  - So Black-Scholes **implied vol** in options and **yield** in bonds.

Stock and Bond Returns

Returns of Stock and Bond and Inflation

Monthly Returns 1942-2014	mean (%)	std (%)	Sharpe ratio	min (%)	max (%)	correlation with Stock	TBill	10Y
Stock (CRSP VW)	1.03	4.16	0.17	-21.58	16.81	1.00	-0.05	0.10
10Y Bond	0.47	2.00	0.08	-6.68	10.00	0.10	0.12	1.00
5Y Bond	0.46	1.38	0.10	-5.80	10.61	0.07	0.19	0.90
2Y Bond	0.42	0.77	0.13	-3.69	8.42	0.08	0.37	0.76
1Y Bond	0.40	0.50	0.16	-1.72	5.61	0.08	0.59	0.62
1M TBill	0.32	0.26		-0.00	1.52	-0.05	1.00	0.12
CPI	0.31	0.45		-1.92	5.88	-0.07	0.26	-0.07
Monthly Returns 1990-2014	mean (%)	std (%)	Sharpe ratio	min (%)	max (%)	correlation with Stock	TBill	10Y
Stock (CRSP VW)	0.87	4.22	0.15	-16.70	11.41	1.00	0.01	-0.06
10Y Bond	0.57	1.99	0.16	-6.68	8.54	-0.06	0.07	1.00
5Y Bond	0.50	1.24	0.20	-3.38	4.52	-0.10	0.15	0.93
2Y Bond	0.39	0.54	0.26	-1.30	2.07	-0.11	0.41	0.74
1Y Bond	0.33	0.31	0.26	-0.33	1.31	-0.03	0.72	0.51
1M TBill	0.25	0.19		-0.00	0.68	0.01	1.00	0.07
CPI	0.21	0.34		-1.92	1.22	-0.04	0.18	-0.16

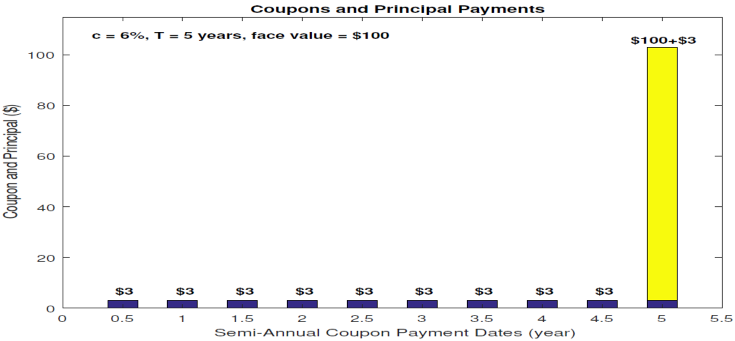
Yield to Maturity and Bond Price

- At issuance, a Treasury bond has the following terms fixed:
  face value $= \$100$; **coupon rate** $= c$; **maturity** $=$ T years.
- Treasury bonds pay coupon semi-annually, and, at issuance, the coupon rate c is chosen so that the bond is priced at par: $P = \$100$ and $c = y$.
- Later, with interest rate fluctuations, both P and y change and there is a *deterministic*, inverse relationship between the two:

$$P = \sum_{n=1}^{2T} \frac{\frac{c}{2} \times 100}{\left(1 + \frac{y}{2}\right)^n} + \frac{100}{\left(1 + \frac{y}{2}\right)^{2T}} .$$

- Increasing interest rate is bad news for bonds and decreasing interest rate is good news for bonds.
- Decreasing interest rate after issuance turns the bond into premium $P > \$100$, and increasing interest rate turns it into discount $P < \$100$.

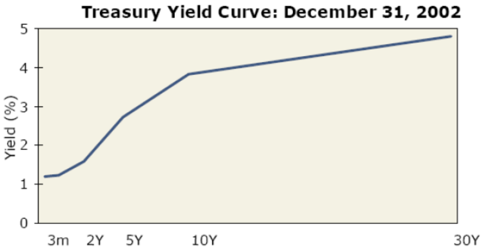
Fixed-Rate Coupon Bonds



$$P = \sum_{n=1}^{2T} \frac{\frac{c}{2} \times 100}{\left(1 + \frac{y}{2}\right)^n} + \frac{100}{\left(1 + \frac{y}{2}\right)^{2T}} .$$

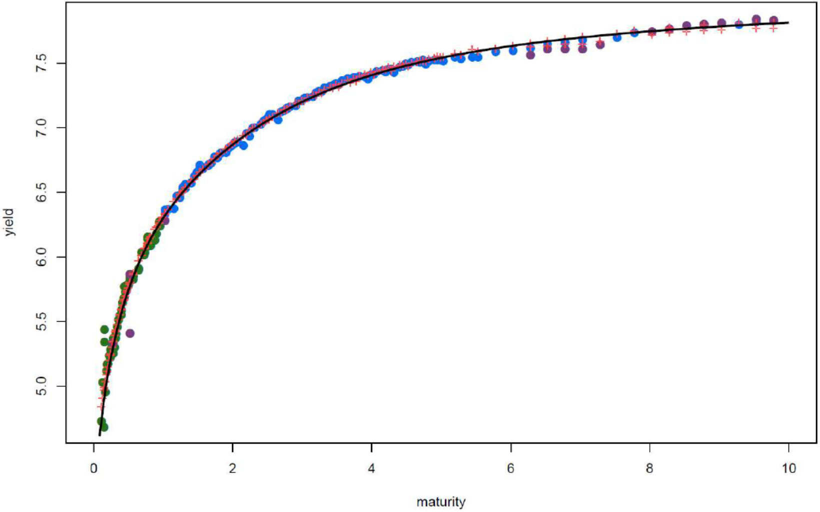
Treasury Yield Curve

- A typical yield curve (also called the term structure of interest rate):

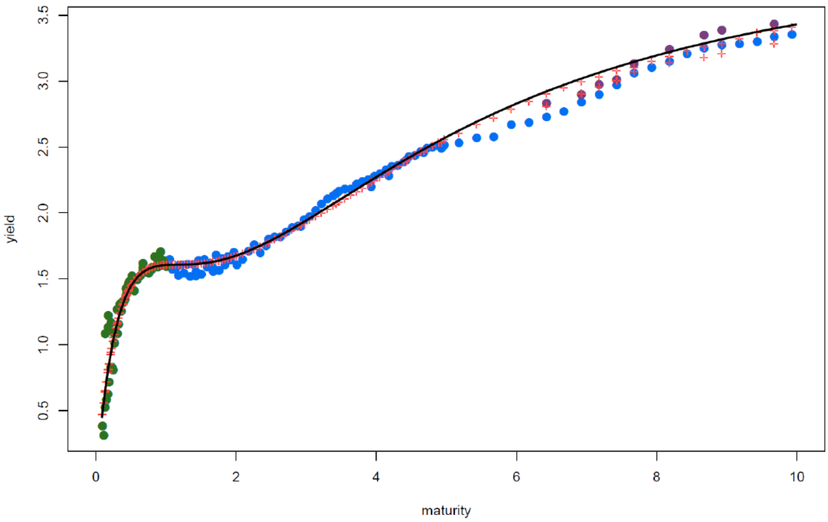


- A yield curve can be created for any specific segment, from triple-A rated mortgage-backed securities to single-B rated corporate bonds.
- The Treasury bond yield curve is the most widely used. The normal shape of the yield curve is upward, but, occasionally, it slopes downward, or inverts.

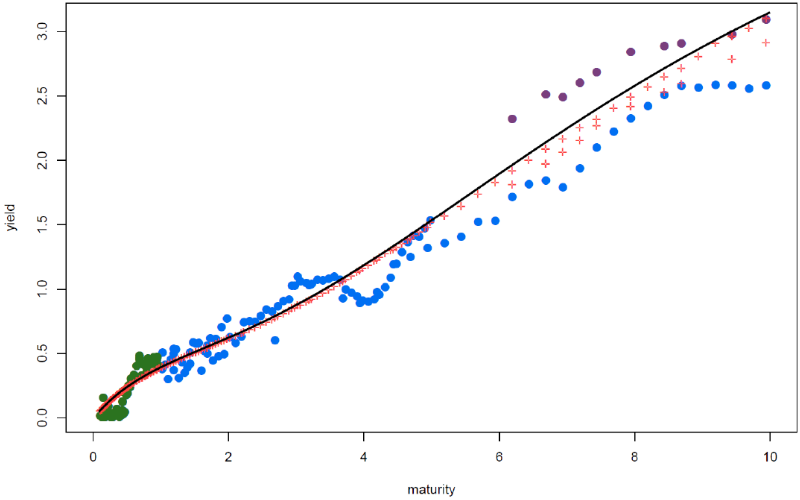
Treasury Yield Curve on November 8, 1994 (Noise=2.60)



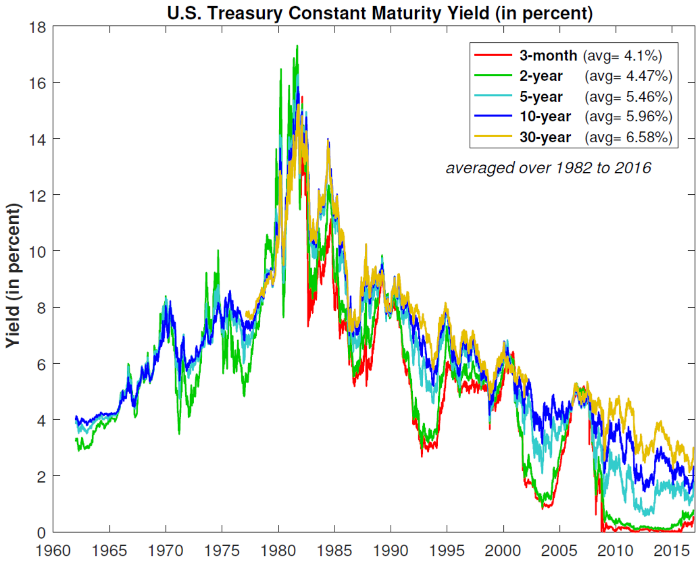
Treasury Yield Curve on September 15, 2008 (Noise=6.64)



Treasury Yield Curve on December 11, 2008 (Noise=20.4)



Treasury Constant Maturity Yields



Daily Changes in Treasury Yields

Daily Changes in Treasury Yields

sample	maturity	std (bp)	min (bp)		max (bp)	
1982-2015	3M	7.63	-104	19820222	169	19820201
	2Y	6.86	-84	19871020	80	19820201
	10Y	6.80	-75	19871020	44	19820201
	30Y	6.30	-76	19871020	42	19820201
1990-2008	3M	5.18	-64	20070820	58	20001226
	2Y	6.05	-54	20010913	36	19940404
	10Y	5.78	-23	19950613	39	19940404
	30Y	4.99	-33	20011031	32	19940404
2008-2015	3M	4.94	-81	20080917	76	20080919
	2Y	4.86	-45	20080915	38	20080919
	10Y	6.42	-51	20090318	24	20080930
	30Y	6.12	-32	20081120	28	20110811

Dollar Duration (DV01) and Modified Duration

- Dollar Duration:

$$-\frac{\partial P}{\partial y} = \frac{1}{1+\frac{y}{2}}\left[\sum_{n=1}^{2T} \frac{n}{2} \times \frac{\frac{c}{2}\times 100}{\left(1+\frac{y}{2}\right)^n} + T \times \frac{100}{\left(1+\frac{y}{2}\right)^{2T}}\right],$$

  which is the negative of \$ change in bond price per unit change in yield.
- DV01 = Dollar Duration/10000 (\$ per 1 basis point change in yield):
- Modified Duration:

$$-\frac{1}{P}\frac{\partial P}{\partial y} = \frac{1}{1+\frac{y}{2}} \frac{\sum_{n=1}^{2T} \frac{n}{2} \times \frac{\frac{c}{2}\times 100}{\left(1+\frac{y}{2}\right)^n} + T \times \frac{100}{\left(1+\frac{y}{2}\right)^{2T}}}{\sum_{n=1}^{2T} \frac{\frac{c}{2}\times 100}{\left(1+\frac{y}{2}\right)^n} + \frac{100}{\left(1+\frac{y}{2}\right)^{2T}}},$$

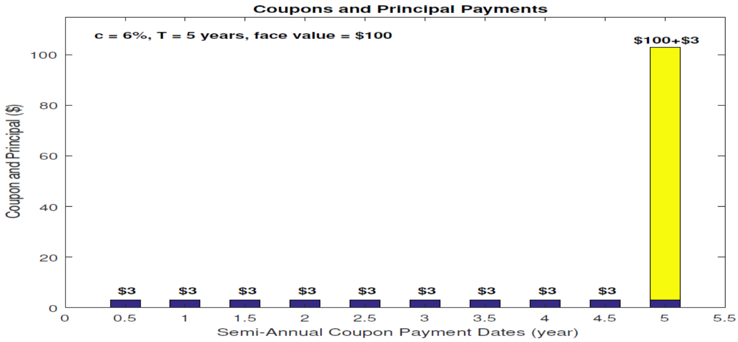
  which is effectively a weighted sum of semi-annual coupon payment dates: $6m$, $1y$, $1.5y$, ..., and T years. It captures the percentage change in bond price (i.e., bond return) per unit change in yield.

Modified Duration

Modified Duration

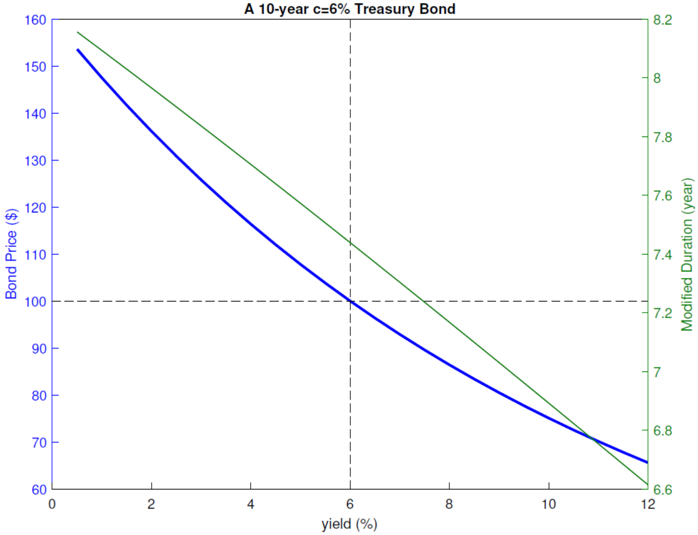
yield y coupon c	2% 2%	5% 5%	6% 4.8%	6% 6%	6% 7.2%	7% 7%	10% 10%
$T = 1$	0.99	0.96	0.96	0.96	0.95	0.95	0.93
$T = 2$	1.95	1.88	1.87	1.86	1.84	1.84	1.77
$T = 3$	2.90	2.75	2.74	2.71	2.68	2.66	2.54
$T = 5$	4.74	4.38	4.36	4.27	4.18	4.16	3.86
$T = 7$	6.50	5.85	5.81	5.65	5.51	5.46	4.95
$T = 10$	9.02	7.79	7.71	7.44	7.21	7.11	6.23
$T = 20$	16.42	12.55	12.12	11.56	11.13	10.68	8.58
$T = 30$	22.48	15.45	14.46	13.84	13.39	12.47	9.46

Calculating Modified Duration



$$D^{\text{mod}} = \frac{1}{1+\frac{y}{2}} \frac{\sum_{n=1}^{2T} \frac{n}{2} \times \frac{\frac{c}{2}\times 100}{\left(1+\frac{y}{2}\right)^n} + T \times \frac{100}{\left(1+\frac{y}{2}\right)^{2T}}}{\sum_{n=1}^{2T} \frac{\frac{c}{2}\times 100}{\left(1+\frac{y}{2}\right)^n} + \frac{100}{\left(1+\frac{y}{2}\right)^{2T}}}$$

Bond Price, Yield, and Duration



Duration and Convexity

- Duration and convexity are meaningful only because we work in the yield space (for convenience), and the profit/loss is in the dollar space.
- Duration is a bridge that connects the two:
  - Dollar Duration:

$$\Delta P_t = P_t - P_{t-1} \approx -\mathrm{D}^{\$} \times (y_t - y_{t-1}) = -\mathrm{D}^{\$} \times \Delta y_t$$

  - Modified Duration:

$$R_t = \frac{\Delta P_t}{P_{t-1}} = \frac{P_t - P_{t-1}}{P_{t-1}} \approx -\mathrm{D}^{\mathrm{mod}} \times (y_t - y_{t-1}) = -\mathrm{D}^{\mathrm{mod}} \times \Delta y_t$$

- The relation between price and yield is not linear, but convex:
  - With decreasing y, duration increases: profits amplified.
  - With increasing y, duration decreases: losses dampened.
- Bonus from positive convexity, not offered by a security linear in y.

Main Takeaways