Abstract

We construct an option-implied Crash Index (CIX) by exploring the pricing difference between the out-of-the-money (OTM) put options and at-the-money (ATM) options, above and beyond a jump-diffusion model (SVJ) that incorporates both stochastic volatility and jump risk and estimated using the joint time-series of the S&P 500 index and ATM options. The construction of our CIX index is analogous to that of the VIX index, except that our focus is on the mean jump size $\mu$ of the SVJ model implied by OTM puts, which are especially sensitivity to crash risk. Empirically, we find that the CIX index is closely related to the non-parametric option-implied skewness, positively correlated with the put/call volume ratio, but unrelated to the VIX index. Post 2008, the CIX index has increased significantly – the mean jump size $\mu$ decreases from the pre-2008 level of -14% to the post-2008 average of -17%. Consistent with the informational channel, we find that, after large increases in CIX, the next-day returns of the S&P 500 index are significantly negative. By contrast, large increases in VIX are followed by large positive returns, indicating a different economic channel.
1 Introduction

Since the seminal work of Black and Scholes (1973) and Merton (1973), the Black-Scholes option pricing model has been extended from its complete-market setting to include additional risk factors – the most prominent of which are stochastic volatility and crash risk. Meanwhile, financial markets since 1973 have witnessed the widespread proliferation of option trading, and, among others, options on the S&P 500 index emerge as an important vehicle for the hedging and speculation of the risk factors embedded in the S&P 500 index. Applying the models to the data, empirical studies on the pricing of S&P 500 index options document the important presence of stochastic volatility and particularly crash risk in the S&P 500 index, allowing the market-traded option prices to help shed light on the single most important equity index in the global market (e.g., Bakshi et al. (1997), Bates (2000), and Pan (2002)).

Our paper builds on this active development of theory and practice since Black and Scholes (1973). Focusing on the crash component of the S&P 500 index, our main objective is to apply the jump-diffusion models to the market-traded option prices to construct a Crash Index (CIX). Our approach is analogous to the construction of the Volatility Index (VIX), which can be traced directly to the volatility parameter \( \sigma \) in the Black-Scholes option model. In fact, the VIX index initially developed by Chicago Board Options Exchange (CBOE) in 1993 was the Black-Scholes volatility implied by the market prices of the 30-day at-the-money (ATM) options. Frequently quoted and monitored as a fear gauge, the VIX index has been the most impactful empirical product of the Black-Scholes model. Likewise, our CIX index is an empirical product of the jump-diffusion models, extracted from the market-traded option prices to capture the crash component of the S&P 500 index.

Central to the class of jump-diffusion models is the jump parameter \( \mu \) first introduced by Merton (1976) to measure the mean jump size conditioning on a Poisson jump arrival. The 1987 stock market crash, when the S&P 500 index dropped by over 20% in just one day, gave this crash parameter \( \mu \) its empirical relevance and importance. Under the risk-neutral measure, a more negative \( \mu \) adds more crash risk, fattens the left tail of the return distribution, and makes the out-of-the-money (OTM) put options more expensive. Just as

\[ \sigma \text{ plays a central role in the pricing of options. As a first-order approximation, the price of an at-the-money (ATM) option is linear in } \sigma, \text{ and the link between the two is such that options traded in the over-the-counter markets are quoted in } \sigma \text{ instead of dollars and centers.} \]
the volatility parameter $\sigma$ in the Black-Scholes model forms the theoretical foundation for the VIX index, the jump parameter $\mu$ in the jump-diffusion models is foundational to our CIX index. Moreover, while the VIX index focuses on the ATM options to extract information with respect to the volatility parameter $\sigma$, our CIX index focuses on the OTM put options to estimate the crash parameter $\mu$.

For the construction of the CIX index, we work with the option pricing model of Bates (2000), which extends the jump-diffusion model of Merton (1976) to incorporate stochastic volatility and allow the jump-arrival intensity to be dependent on the latent stochastic volatility. For brevity, we refer to our model as the stochastic volatility model with jump (SVJ). Unlike the Black-Scholes model, the estimation of the SVJ model is more involved as it contains a latent state variable and an array of model parameters that govern the joint dynamics of the stock prices and stochastic volatility and the market prices of the risk factors. Following Pan (2002), we use the joint time-series of the S&P 500 index and options to simultaneously estimate the model parameters and the time-series of the state variable (i.e., the latent volatility). On each trading day, the latent volatility is a function of the ATM option price observed on that day and the unknown model parameters including the mean jump size $\mu$. We then estimate the model parameters using the moment conditions constructed from the joint dynamics of stock prices and stochastic volatility.

Equipped with the model estimation, we construct the Crash Index by exploring the pricing discrepancy between OTM puts and ATM options above and beyond the SVJ model. On each trading day, we plot a crash curve (i.e., option-implied $\mu$) using options of the same time to expiration but differing strike prices. This is analogous to the volatility curve (i.e., option-implied $\sigma$), but instead of using the Black-Scholes model to estimate the option-implied $\sigma$ across the varying strike prices, we use the SVJ model to estimate the option-implied $\mu$ across strike price, while keeping the other model parameters and the estimated latent state variable fixed. Central to our CIX index is the difference between the crash parameter $\mu$ implied by the OTM puts and that implied by the ATM option.\(^3\)

As an intuitive illustration of what is captured by our CIX index, we can go back to the original Black-Scholes model. If the market-traded options are priced according to the Black-Scholes model, then the volatility curve would be flat. The fact that the volatility curve is not flat calls for the class of jump-diffusion models such as the SVJ model to incorporate stochastic volatility and crash risk. Likewise, within the context of our SVJ model, if the

\(^3\)It is worth noting that our methodology for implying crash risk adjusts for the impact of stochastic volatility, thereby distinguishing between jump crash risk and volatility as discrete contributors to the pricing discrepancy between OTM and ATM options. Our methodology also allows the jump arrival intensity to be dependent on the stochastic volatility — an empirical fact document by Pan (2002) to be important to reconcile the joint dynamics of the S&P 500 index and options. Our empirical findings corroborate such distinctions, revealing markedly different impacts of the CIX and VIX on asset prices.
market-traded OTM puts and ATM options price in the same mean jump size $\mu$, then the crash curve would be flat. Conversely, by exploring the difference in $\mu$ implied by OTM puts and ATM options, we zero in on the unique crash information, if any, embedded in the pricing of the OTM puts. Given their sensitivity to tail events, the OTM puts on the S&P 500 index are among the most actively traded index options used by investors to hedge and speculate on the crash risk. By focusing on such options, our CIX index is designed to extract the crash risk anticipated or priced by such investors.

Our empirical results can be summarized as follows. First, applying the estimated SVJ model to the short-dated OTM puts to back out the respective option-implied crash parameter $\mu$, we find that, relative to the ATM-implied $\mu$, such option-implied crash parameters are consistently more negative, particularly for the most actively traded OTM puts with strike prices ranging from 95% to 98% of the spot price. This pattern aligns with the expectation that the most frequently traded OTM put options would be highly sensitive to anticipated future crashes and motivates us to construct CIX as an average of $-\mu$ implied by OTM puts, focusing on strike prices within the most informative range. Considering that the disparity between OTM puts and ATM options is influenced by both the volatility level and $\mu$, our use of the SVJ model in constructing the CIX index allows us to take out the volatility impact and isolate the effect of $\mu$. From this perspective, our CIX index is a more precise measure of crash risk implied by OTM puts.\(^4\)

Second, utilizing the non-parametric approach of Breeden and Litzenberger (1978), we construct a skew index, previously employed by Bakshi et al. (2003) to analyze cross-sectional option pricing. Our CIX index undergoes further validation through this non-parametric measure of option-implied skewness: assuming the data-generating process aligns with our SVJ models, the skewness should be primarily driven by the level of $\mu$ and the jump arrival intensity. Indeed, we observe a significantly positive relationship between CIX and option-implied skewness. We also find that the skewness index is influenced by VIX, consistent with our model’s specification that the jump arrival intensity is linear in the volatility level. By contrast, our CIX index is found to be uncorrelated with the VIX index, consistent with its focus on the crash parameter $\mu$. While both measures are complimentary to each other, the parametric approach has the advantage to separately model and estimate the effects of volatility and crash risk. Moreover, while the non-parametric approach relies on the entire collection of options to estimate the skewness, our approach can estimate the crash parameter for each option, allowing us to focus on those contracts that are informationally richer.

Third, we extend our analysis to the time-series dynamics of CIX, linking it to the

\(^4\)Compared with their counterparts in the $\sigma$ space, the difference in option-implied volatility between OTM puts and ATM options can be driven by the presence of stochastic volatility and the stochastic jump arrival intensity.
divergence between OTM and ATM options. We quantify the time-series of this divergence by the disparity in Black-Scholes implied volatility between ATM options and the average of OTM options, within the same range of strike prices used to formulate CIX. As anticipated, this implied volatility spread is elucidated by both CIX, representing crash risk, and VIX, symbolizing volatility risk. Moreover, we discover that CIX is intimately correlated with non-parametric option-implied skewness and the put/call option trading volume ratio. In alignment with our expectation that CIX does not associate with VIX as an independent risk factor, it also remains unconnected to other risk factors encapsulated in macroeconomic variables, such as treasury term spread, corporate bond default spread, etc. An intriguing observation is that our CIX increased significantly after the 2008 global financial crisis, with the mean CIX rising from 14% to 17%, indicating a perceptible shift in crash risk anticipation subsequent to the crisis.

Finally, using the CIX index to predict stock market returns, we find that a sudden increase in CIX forecasts a negative S&P 500 index return on the next day. This predictability is distinct from that of the VIX index, as presented in Hu et al. (2022), where a sizable surge in the VIX often signals positive returns for the S&P 500 index. We observe that an extreme surge in CIX precedes a significant decline in the S&P 500 index return by 48 basis points, a sharp contrast to the overall sample average of 3-4 basis points. Conversely, our findings show that a surge in VIX is associated with a marked upswing in the stock market return by 59.88 basis points. This contrast between the CIX and VIX in their relationship with subsequent stock market performance emphasizes the importance of distinguishing jump risk from stochastic volatility risk, a central theme our paper explores. Further, we employ a predictive regression framework to control for the VIX’s surging effect and identify a negative predictability, particularly concentrated at times when unanticipated shocks impact the CIX. This predictability remains consistent across various methods we use to assess the shocks to CIX.

In conclusion, our empirical research underscores a robust inverse correlation between the Crash Index (CIX) and future stock market returns. This pronounced link is particularly evident during episodes of abrupt and substantial CIX fluctuations, underscoring the vital role of heightened forward-looking crash risk as a key short-term factor in asset pricing dynamics.

The rest of the paper is organized as follows. Section 2 presents our SVJ model and the according model estimation. Section 3 illustrate our construction of the CIX index. Section 4 delves into empirical results of testing the implication of CIX. Section 5 concludes.

Related Literature – Our research belongs to the classical option pricing literature, beginning with Black and Scholes (1973) and Merton (1973) and evolving with the works of Merton (1976), Cox and Ross (1976), and Cox et al. (1979). Our work builds on the explo-
ration of models with stochastic state variables, such as stochastic volatility and tail risks, as seen in Heston (1993), Duffie et al. (2000), Bates (2000), Bakshi et al. (1997), and Pan (2002).

As an empirical study on the option pricing model, our paper intersects with the literature focused on the estimation methodologies of stochastic volatility models and state-dependent jump models. Chernov and Ghysels (2000) apply an indirect inference approach to stochastic volatility models, while Eraker et al. (2003) employs a Markov chain Monte Carlo method to jointly estimate jumps and stochastic volatility. Furthermore, alternative models such as Bates (2006), and Christoffersen et al. (2012), assume that volatility follows a GARCH-class dynamic, which enables a filtering approach to estimate volatility and subsequently separate the jumps. Our empirical estimation differs from these approaches by adopting the joint time-series approach of Pan (2002).

Our research also aligns with the asset pricing literature that derives non-parametric measures of jump or crash risk from option data. Given the negative skewness of stock returns, Bakshi et al. (2003) utilized the insights of Breeden and Litzenberger (1978) to conceive an option-implied skewness measure. Cremers et al. (2015) construct a vega-neutral option portfolio to approximate stock return jumps and use cross-sectional stocks to test its market price of risk. By taking advantage of the parametric jump-diffusion model, our paper differs from the non-parametric approach commonly found in literature when identifying jump risk. In relation to the non-parametric estimation of higher moments (i.e., the skewness), our approach allows us to separately identify the contributions of stochastic volatility and jump risk.

Finally, our work is associated with asset pricing models that scrutinize how investors price crashes or rare disasters within a structural framework (for instance, Liu et al. (2005), Gabaix (2012), Wachter (2013)). Unlike these studies, which calibrate a static model for jump risks, our paper requires a tractable framework to develop a measure of dynamic jump risk and/or jump-risk premiums.

2 Model and Model Estimation

2.1 The Data Generating Process

In this study, we employ the same model of stock return dynamics as outlined in Bates (2000) and Pan (2002). The stock price $S_t$ over time $t$ follows the data-generating process delineated below,
\[dS_t = [r_t - q_t + \eta S_t + \lambda V_t (\mu - \mu^*)] S_t \, dt + \sqrt{V_t} S_t \, dW_t^{(1)}
+ dZ_t - \mu S_t \lambda V_t \, dt\]  
\[dV_t = \kappa_v (\bar{v} - V_t) \, dt + \sigma_v \sqrt{V_t} \left( \rho dW_t^{(1)} + \sqrt{1 - \rho^2} \, dW_t^{(2)} \right),\]

where \(r\) denotes the interest-rate process, \(q\) signifies the dividend yield, \(W = [W^{(1)}, W^{(2)}]^{\top}\) constitutes an adapted standard Brownian motion in \(\mathbb{R}^2\), and \(Z\) represents a pure-jump process.\(^5\)

Supported by literature, this model underscores two crucial attributes: it fits the contemporaneous dynamic of stock returns and option prices, and it ensures tractability for solving option price and estimating a parsimonious set of parameters. Firstly, stochastic volatility in Eq. (2) drives the stock return dynamic as a latent variable. The correlation coefficient \(\rho\) encapsulates the characteristic of this process, i.e., stock returns are typically negatively correlated with volatility fluctuations. More specifically, \(V_t\) is a one-factor "square-root" process, characterized by a stable long-term mean \(\bar{v}\), mean-reversion rate \(\kappa_v\), and volatility coefficient \(\sigma_v\).\(^6\)

Secondly, jump risk in the stock price \(Z_t\) features as a state-dependent jump intensity \(\lambda V_t\). Here, \(Z\) consists of two elements: random jump-event times and random jump sizes. The jump-event times \(\{T_i : i \geq 1\}\) are dictated by a state-dependent stochastic intensity process \(\{\lambda V_t : t \geq 0\}\) for some non-negative constant \(\lambda\). At the \(i\)th jump event, the stock price jumps from \(S(T_i^-)\) to \(S(T_i^-) \exp( U_s^i)\), where \(U_s^i\) is normally distributed with mean \(\mu_J\) and variance \(\sigma_J^2\), independent of \(W\), inter-jump times, and \(U_j^j\) for \(j \neq i\). The conditional probability at time \(t\) of a jump prior to \(t + \Delta t\) is approximately \(\lambda V_t \Delta t\) for a small \(\Delta t\). When a jump event occurs, the mean relative jump size is \(\mu = E(\exp(U_s^i) - 1) = \exp(\mu_J + \sigma_J^2/2) - 1\). Combining the effects of random jump timing and sizes, the final term \(\mu S_t \lambda V_t \, dt\) in Eq. (1) offsets the instantaneous change in expected stock returns introduced by the pure-jump process \(Z\).

Extending the existing literature that employs this or related data-generating processes for explicating the joint dynamic of stock and option prices, our research deviates to explore a unique economic question. This paper specifically delves into the option-implied equity risk

\(^5\)Unlike Pan (2002), we treat the interest rate and dividend yield as time-varying constants for simplicity. This approach doesn’t hinder our ability to fit the short-term option price. In empirical tests, we update the interest rate and dividend yield using daily data.

\(^6\)Note that we use the term "volatility" to denote variance \(V\), often seen as the standard deviation of returns. This terminology shift should not lead to confusion.
premium from the jump process, encapsulated by the term \( \lambda V_t (\mu - \mu^*) \) in Eq. (1), where \( \mu \) is the jump size in the stock price dynamic and \( \mu^* \) is the jump size under a risk-neutral measure. Assuming a known \( V_t \) implied from option prices, \( \mu - \mu^* \) captures a market price of jump risk, distinguishing itself as a different source of risk from volatility. Therefore, our study poses the question: given the same level of volatility, what ramifications does a heightened priced jump risk entail? To focus on this question, we take the \( -\mu^* \) as the key parameter to capture the jump risk premium since the jump size \( \mu \) under physical measure is small and \( \mu^* \) reflects how investors price the downside risk.

Recognizing a comprehensive literature on the impact of option-implied volatility, our approach distinctly bifurcates the effects of the option-implied volatility and jump size \( \mu^* \). Hence, our methodology underscores the novelty of our research question while maintaining alignment with the existing theoretical structure. Interestingly, we observe distinct behavior of our jump risk measure compared to the implied volatility, regardless of whether it’s measured by our implied-state approach or model-free approach prevalent in the literature. This difference is discussed later in this section with relevant examples.

To set the stage for our subsequent analysis on option-implied jump size, we will discuss the price of risk, as represented by the equity risk premium formula in Eq. (1), and elaborate on the derivations of option pricing in the following sections.

2.2 The Market Prices of Risks

The model adopted in our study does not guarantee a complete market with respect to the risk-free bank account, the underlying stock, and a finite number of options contracts. This is particularly due to the random jump size in the stock price dynamics. For our research objectives, we employ a plausible pricing kernel that accommodates the three primary sources of risk: diffusive price shocks, jump risks, and volatility shocks.

For clarity, this section presents the ”risk-neutral” price dynamics as defined by our selected pricing kernel.\(^7\) Let \( Q \) be the equivalent martingale measure associated with our selected pricing kernel. Under \( Q \), the dynamics of \((S,V)\) are expressed as:

\[
\text{d}S_t = \left[ r_t - q_t \right] S_t \, dt + \sqrt{V_t} S_t \, dW_t^{(1)}(Q) + \text{d}Z_t^Q - \mu^* S_t \lambda V_t \, dt
\]

\(^7\)The pricing kernel we apply is the same as derived in the Appendix A of Pan (2002). Specifically, the jump sizes \( U^\pi_t \) are assumed to be i.i.d. normal with mean \( \mu_\pi \) and variance \( \sigma_\pi^2 \), and are assumed to be independent of the Brownian motions, and inter-jump times. We enforce the constraint that the mean relative jump size in the state-price density to be zero. That is, \( \mu_\pi + \sigma_\pi^2/2 = 0 \). This constraint is, in fact, translated to a zero jump-timing risk premium.
\[dV_t = [\kappa_v (\bar{v} - V_t) + \eta^V V_t] \, dt + \sigma_v \sqrt{V_t} \left( \rho dW_t^{(1)}(Q) + \sqrt{1 - \rho^2} \, dW_t^{(2)}(Q) \right), \] (4)

where \(W(Q) = [W^{(1)}(Q), W^{(2)}(Q)]\) represents a standard Brownian motion under \(Q\). [A formal definition of \(W(Q)\) is provided in Appendix A.] The pure-jump process \(Z^Q\) has a distribution under \(Q\) identical to the distribution of \(Z\) under \(P\), as defined in Eq. (1), except that under \(Q\), the jump size \(\mu^*\) accommodates a risk premium for jump uncertainty. With all other factors equivalent to the physical measure dynamic, the risk-neutral mean relative jump size is \(\mu^* = E^Q (\exp (U^s) - 1) = \exp (\mu_j + \sigma_j^2/2) - 1\). Echoing the discussion regarding the data-generating process, we observe that the final term \(\mu^* S_t \lambda V_t \, dt\) in Eq. (2.4) serves as a compensator for the pure-jump process \(Z^Q\) under the risk-neutral measure.

Our specification of the risk-neutral dynamics of \((S, V)\) facilitates an intuitive understanding of the pricing of various risk factors. Focusing first on the market prices of jump risks, we see that by permitting the risk-neutral mean relative jump size \(\mu^*\) to deviate from its data-generating counterpart \(\mu\), the time- \(t\) expected excess stock return compensating for jump-size uncertainty is \(\lambda V_t (\mu - \mu^*)\). In light of the widely accepted fact that stock returns are negatively skewed, the physical jump size is typically negative. Moreover, in our specification, the risk of a sharp decline, or "crash risk," is positively priced in the equity return, as captured by a more negative jump size under the \(Q\) measure, i.e. \(\mu - \mu^* > 0\).

"Conventional" return risks, or "Brownian" shocks, carry premiums parameterized by \(\eta^V V_t\) for a constant coefficient \(\eta^V\). This is similar to the risk-return trade-off in the CAPM framework. Premiums for "volatility" risks, however, are less transparent, given that volatility is not directly tradable. Due to the inherent volatility of volatility itself, options may reflect an additional volatility risk premium. Volatility risk is priced via the supplementary term \(\eta^V V_t\) in the risk-neutral dynamics of \(V\) in Eq. (4). A positive coefficient \(\eta^v\) implies that the time- \(t\) instantaneous mean growth rate of the volatility process \(V\) is \(\eta^V V_t\) higher under the risk-neutral measure \(Q\) than under the data-generating measure \(P\). Given that option prices respond positively to the volatility of the underlying price in this model, option prices rise with \(\eta^v\).

### 2.3 Option Pricing

Our paper simplifies the option pricing solution compared to Pan (2002) by assuming that the interest rate and dividend yield are time-varying constants. This simplification, as pointed

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8Our specification primarily focuses on the risk premium for jump-size uncertainty, while overlooking the risk premium for jump-timing uncertainty by assuming \(\lambda^* = \lambda\). This implies that all jump risk premiums are subsumed by the jump-size risk premium coefficient \(\mu - \mu^*\). We adopt this specification largely for empirical convenience, as identifying the term \(\lambda^* \mu^*\) can be cumbersome.
out in the literature, does not impact our analysis since the stochastic dynamics of the interest rate and dividend yield play a minor role in fitting option prices, especially for short-term options. We denote the set of the model parameters as:

\[ \vartheta = (\kappa_v, \bar{v}, \sigma_v, \rho, \lambda, \mu, \sigma_J, \eta^s, \eta^v, \mu^*) \]  

(5)

Let \( C_t \) represent the price at time \( t \) of a European-style call option on \( S \), with a strike price of \( K \) and an expiration date at \( T = t + \tau \). By taking advantage of the affine structure of \((\ln S, V, r, q)\) and utilizing the transform-based approach (refer to, for example, Heston (1993), Bakshi et al. (1997), Bakshi and Madan (2000), Duffie et al. (2000)), we can express \( C_t \) as follows:

\[ C_t = E^Q_t \left[ \exp \left( -\int_t^T r_u \, du \right) (S_T - K)^+ \right] = S_t f \left( V_t, \vartheta, r_t, q_t, \tau, \frac{K}{S_t} \right) \]  

(6)

We derive an explicit formulation for \( f \) and the relating numerical calculation in Appendix A.

2.4 Estimation

In this section, we present the method for estimating the parameters in Eq. (1), (2) by employing time-series data of S&P 500 index’s spot and option prices \( \{S_t, C_t\} \). Our model, indicated in Eq. (6), offers analytical tractability, demonstrating the joint dynamics of spot and option prices through two state variables, \((S, V)\). We utilize the "implied-state" generalized method of moments (IS-GMM) approach, as outlined in Pan (2002). This technique transforms observed option prices to imply the latent variable \( V_t \) as if it were directly observed, and then estimates the parameters in Eq. (5) by matching the moments condition of \((S, V)\).

We initially define the log return from time \( t - \Delta t \) to \( t \) as:

\[ y_t = \ln S_t - \ln S_{t-\Delta t} - r_t - q_t, \]  

(7)

where \( \Delta t \) represents a short time interval. For practical purposes, we employ daily frequency data as a proxy for a short time interval of a year, with \( r_t, q_t \) derived from the same frequency data. To streamline notation, we designate the short time interval \( \Delta t \) as one unit of time, replacing \( t - \Delta t \) with \( t - 1 \) throughout the remainder of the paper.

Additionally, we extract the stochastic volatility \( V_t \) from a near to at-the-money call option with around 30 days to maturity from \( C_t = S_t f \left( V_t, \vartheta, r_t, q_t, \tau, \frac{K}{S_t} \right) \),
\[ V_t = g \left( \frac{C_t}{S_t}, \vartheta, r_t, q_t, K \frac{S_t}{S_t} \right), \quad (8) \]

where we select the moneyness and maturity such that \( \frac{K}{S_t} \approx 1, \tau \approx 30/365 \). Leveraging Eq. (7) and (8), we can express the joint dynamics of \( y_t, V_t \) as a function of the time-series data of the S&P 500 index’s spot and option prices \( \{S_t, C_t\} \) and the parameter set \( \vartheta = (\kappa, \bar{\nu}, \sigma, \rho, \lambda, \mu, \sigma_J, \eta, \eta^p, \mu^*) \).

The IS-GMM methodology depicts the joint moments of \( (y_t, V_t) \), transformed from the time-series data \( \{S_t, C_t\}_{t=1,...,T} \), as a function of parameters in \( \vartheta \) and the latent variable \( V \). Consistent with Pan (2002), we employ seven conditional moments of \( (y_t, V_t) \) at time \( t - 1 \). We designate \( M_1 (V_{t-1}, \vartheta) = \mathbb{E}_{t-1}^\vartheta (y_t), M_2 (V_{t-1}, \vartheta) = \mathbb{E}_{t-1}^\vartheta (y_t^2), M_3 (V_{t-1}, \vartheta) = \mathbb{E}_{t-1}^\vartheta (y_t^3), \) and \( M_4 (V_{t-1}, \vartheta) = \mathbb{E}_{t-1}^\vartheta (y_t^4) \) to denote the first four conditional moments of return. We label \( M_5 (V_{t-1}, \vartheta) = \mathbb{E}_{t-1}^\vartheta (V_t) \) and \( M_6 (V_{t-1}, \vartheta) = \mathbb{E}_{t-1}^\vartheta (V_t^2) \) as the first two conditional moments of volatility. Lastly, we allow \( M_7 (V_{t-1}, \vartheta) = \mathbb{E}_{t-1}^\vartheta (y_t V_t) \) to denote the first cross moment of return and volatility. We commence with the following moment conditions:\(^9\)

\[ \mathbb{E}_{t-1}^\vartheta (\varepsilon_t) = 0, \quad \varepsilon_t = [\varepsilon_{t}^{y1}, \varepsilon_{t}^{y2}, \varepsilon_{t}^{y3}, \varepsilon_{t}^{y4}, \varepsilon_{t}^{v1}, \varepsilon_{t}^{v2}, \varepsilon_{t}^{yw}]^\top, \quad (9) \]

where

\[ \begin{align*}
\varepsilon_{t}^{y1} &= y_t - M_1 (V_{t-1}, \vartheta), \\
\varepsilon_{t}^{y2} &= y_t^2 - M_2 (V_{t-1}, \vartheta), \\
\varepsilon_{t}^{y3} &= y_t^3 - M_3 (V_{t-1}, \vartheta), \\
\varepsilon_{t}^{y4} &= y_t^4 - M_4 (V_{t-1}, \vartheta), \\
\varepsilon_{t}^{v1} &= V_t - M_5 (V_{t-1}, \vartheta), \\
\varepsilon_{t}^{v2} &= V_t^2 - M_6 (V_{t-1}, \vartheta), \\
\varepsilon_{t}^{yw} &= y_t V_t - M_7 (V_{t-1}, \vartheta).
\end{align*} \]

To identify the ten parameters in \( \vartheta \), we introduce three instrument moments using the lagged value \( V_{t-1} \),

\[ \begin{align*}
\mathbb{E}_{t-1}^\vartheta (\varepsilon_t V_{t-1}) &= 0, \\
\mathbb{E}_{t-1}^\vartheta (\varepsilon_t^2 V_{t-1}) &= 0, \\
\mathbb{E}_{t-1}^\vartheta (\varepsilon_t^{yw} V_{t-1}) &= 0. \quad (10)
\end{align*} \]

Besides using the moment conditions from \( (y_t, V_t) \), we further incorporate one in-the-money (ITM) call option with maturity around 30 days and one at-the-money (ATM) call option with maturity around 60 days to better identify the jump risk and the account for long-term information. We form two extra moment conditions to minimize the percentage pricing error.

\(^9\)A recursive formula to calculate the conditional moments of \( (y_t, V_t) \) is derived in Pan (2002).

\(^{10}\)The moment condition we select here is different from Pan (2002), which applies an efficient transform of the seven moments using a conditional instruments proposed by Hansen (1985). This method provides more efficiency for estimation yet requires more computing time for numerical derivatives. Since we have more observations nowadays, we apply our current method to save computing time and relies on the larger sample to reduce the standard error of parameters.
of our model on these two options.\footnote{We choose in-the-money (ITM) calls instead of out-the-money (OTM) since the ITM calls has higher magnitude of price, which helps us to get stable optimization convergence.} Suppose the observed market prices of these two options at time $t$ is $C_{t}^{60,\text{ATM}}$ and $C_{t}^{30,\text{ITM}}$, and the theoretical prices using our model given the same set of state variable and the parameter set $\vartheta$ is $C_{t,\text{model}}^{60,\text{ATM}}(\vartheta)$ and $C_{t,\text{model}}^{30,\text{ITM}}(\vartheta)$, we can write the moment conditions as,

$$E_{t-1}^{\vartheta}(\varepsilon_{t}^{30,\text{ITM}}) = 0, \quad \varepsilon_{t}^{30,\text{ITM}} = \frac{C_{t,\text{model}}^{30,\text{ITM}}(\vartheta) - C_{t}^{30,\text{ITM}}}{C_{t}^{30,\text{ITM}}},$$

$$E_{t-1}^{\vartheta}(\varepsilon_{t}^{60,\text{ATM}}) = 0, \quad \varepsilon_{t}^{60,\text{ATM}} = \frac{C_{t,\text{model}}^{60,\text{ATM}}(\vartheta) - C_{t}^{60,\text{ATM}}}{C_{t}^{60,\text{ATM}}}.$$  \hspace{1cm} (11)

Together with Eq. (9), (10) and (11), we have a system of 12 moment conditions to identify the 10 parameters. We stack the vector $h_t$ across $t = 1...T$ and compute the sample average of the moment conditions as,

$$G_{T}(\vartheta) = \frac{1}{T} \sum_{n \leq N} h(y_t, V_{t,\vartheta}, \vartheta).$$

The IS-GMM estimator is hence,

$$\hat{\vartheta} = \arg \min_{\vartheta \in \Theta} G_{T}(\vartheta)^{T}W_{T}G_{T}(\vartheta),$$

where $W$ is the weighting matrix, which is usually selected as the inverse of the covariance matrix of $G_{T}(\vartheta)$.

\section{2.5 Data}

The S&P 500 index option and spot prices used to estimate our model and construct the CIX are obtained from the OptionMetrics database. Our sample spans from January 1996 to December 2021. We calculate the price of each option as the average of its bid and ask price. Excluded from our analysis are options with zero open interest, zero bid prices, and missing implied volatility or delta. The latter typically arises for options with non-standard settlement or for options with an intrinsic value exceeding the current mid-price. We also utilize the daily composite dividend yield data for the S&P 500 from OptionMetrics, as well as daily interest rate data interpolated from zero coupon certificate of deposit rates. Daily index return data is sourced from the Center for Research in Security Prices (CRSP). Besides the option price data, we further use the volatility surface data interpolated by
OptionMetrics to check the robustness of our model estimation and CIX index.

To investigate the factors that influence variations in our CIX measure, we acquire daily VIX data from the Chicago Board Options Exchange (CBOE) website. Furthermore, we obtain macroeconomic variables such as the term spread and default spread from the Federal Reserve Economic Data (FRED) website.

3 Option-Implied Crash Index

3.1 Infer Crash Risk from Option Prices

Our approach uses several options at each time of our entire sample to infer the stochastic volatility \( V_t \) and estimate the parameters \( \vartheta = (\kappa_v, \bar{v}, \sigma_v, \rho, \lambda, \mu, \sigma_J, \eta^s, \eta^v, \mu^*) \). In practice, the jump size implied from options beyond our estimation procedure may differ across varying strike prices and maturities. This variation of implied jump size offers valuable insights into the risk-neutral distribution, particularly the crash probability encapsulated by deep out-of-the-money put options. Hence, we extend our estimation while holding all other parameters \( \vartheta^\perp = (\kappa_v, \bar{v}, \sigma_v, \rho, \lambda, \mu, \sigma_J, \eta^s, \eta^v) \) and the stochastic volatility \( V_t \) constant, to derive a jump size \( \mu^*_t(\tau_i, K_i) \) specific to strike price \( K_i \) and maturity \( \tau_i \) for all observed options \( i = 1...N(t) \) at time \( t \). More precisely, for any given option price (scaled by the spot price) at time \( t \),

\[
\mu^*_t(\tau_i, K_i) = g^\mu \left( f(V_t, \vartheta^\perp, \mu^*, r_t, q_t, \tau_i, K_i / S_t) \right).
\]  

(12)

Our methodology bears resemblance to the vast literature on the "volatility surface", which uses the Black-Scholes model as a benchmark with a constant volatility and extends it to infer \( \sigma(\tau_i, K_i) \) across options with different maturities and strike prices. However, in our case, we consider the state-dependent jump diffusion model as a benchmark with a constant jump size \( \mu^* \) and extend it to infer a complete surface of crash risk priced across all options.

Leveraging the strengths of the implied-state estimator, our method effectively distinguishes between the pricing impacts of stochastic volatility and jump size. Additionally, as verified in related literature, our benchmark state-dependent jump diffusion model effectively captures option pricing. Consequently, our extension to imply jump sizes should accurately reflect the dynamic variation of market crash risk pricing and capture the conditional information of investors’ forward-looking crash expectations.
Our model produces a broad set of implications. For instance, one could analyze the pattern of \( \mu^* (\tau, K_i) \) across various strike prices to explore a concept similar to the "implied volatility smirk" frequently discussed in the existing literature—what we might term a "jump risk premium surface." However, this paper primarily focuses on the temporal changes in the jump risk premium and how these fluctuations impact the stock market. In line with this focus, we introduce the Crash Index (CIX)—an index that encapsulates the average jump risk premiums derived from short-term, out-of-the-money options.

### 3.2 Construct the Crash Index (CIX)

The calculation of the Crash Index (CIX) is conceptually similar to that of the Volatility Index (VIX). Each day, we select two expiration dates such that their maturities are the closest to 30 days, with \( \tau_1 < 30 < \tau_2 \). We then concentrate on the out-of-the-money (OTM) put options expiring on these two dates. For each expiration date, we select OTM put options with a strike price approximately 95% of the current spot price and compute the average implied jump risk premium

\[
\mu^*(\tau_{i=1,2}) = \frac{1}{N_i} \sum_{s=1}^{N_i} \mu(\tau_i, K_s), \quad \text{s.t. } K_s / S \in [0.93, 0.97].
\]

We then interpolate the implied jump size at the two expiration dates to construct a 30-day measure of forward-looking jump risk premium.

\[
CIX_t = -\mu_t(\tau_1) \ast \frac{\tau_2 - 30}{\tau_2 - \tau_1} - \mu_t(\tau_2) \ast \frac{30 - \tau_1}{\tau_2 - \tau_1}
\]  

(13)

This methodology can also be applied to approximate a crash risk index for a forward-looking period that’s shorter or longer than 30 days. We found that all such indexes exhibit similar dynamics.\(^\text{13}\) However, we choose the 30-day measure as our primary output, as it aligns with the conventions in the literature and bears resemblance to the construction of the VIX. Specifically, the VIX index is formulated as

\[
VIX_t = 100 \ast \sqrt{\sigma^2_t(\tau_{i=1,2}) \ast \frac{\tau_2 - 30}{\tau_2 - \tau_1} + \sigma^2(\tau_2) \ast \frac{30 - \tau_1}{\tau_2 - \tau_1}},
\]

(14)

where \( \sigma^2_t(\tau_{i=1,2}) \) represents the non-parametric second moment implied from option prices.\(^\text{14}\)

\(^\text{12}\)At the beginning of our sample period, the OTM options in this range may not be available as we filtered out some illiquid prices. In such cases, we average put options such that \( K / S \in [0.9, 1] \)

\(^\text{13}\)In addition, we test CIX using a volume-weighted jump size. The result is highly similar.

\(^\text{14}\)This methodology can be checked on the CBOE website: https://cdn.cboe.com/api/global/us_indices/governance/Volatility_Index_Methodology_Cboe_Volatility_Index.pdf
The non-parametric moments method stems from literature using OTM options to infer the risk-neutral distribution of asset returns (see derivations in Breeden and Litzenberger (1978), and density estimation, e.g., Aït-Sahalia and Lo (1998)). Besides the second moments presented by VIX, the Crash Index is more pertinent to the third moments, or skewness implied from option prices, as measured by Bakshi et al. (2003). Within our parametric framework, both the level of volatility and the jump size parameter \( \mu^* \) drive the level of skewness, motivating us to isolate the role of crash risk by CIX in addition to using the skewness index to test asset pricing implications. We derive the second and third moments under SVJ models to harmonize our methods with the non-parametric measure in the subsequent section.

### 3.3 Relation to the Non-Parametric Approach

We firstly present the construction of risk neutral moments in Bakshi et al. (2003) and then reconcile with our parametric framework. Let the \( \tau \)-period return at time \( t \) be given by the log price relative:

\[
R(t, \tau) = \ln S_{t+\tau} - \ln S_t.
\]

The risk neutral moments are,

\[
V(t, \tau) = E_t^Q \left[ \exp \left( -\int_t^T r_u \, du \right) \left( R(t, \tau) - E_t^Q [R(t, \tau)] \right)^2 \right],
\]

\[
W(t, \tau) = E_t^Q \left[ \exp \left( -\int_t^T r_u \, du \right) \left( R(t, \tau) - E_t^Q [R(t, \tau)] \right)^2 \right].
\]  

The second moments is equivalent to \( \sigma^2_t(\tau) \) used in the VIX construction. And the skewness index (\( SK_t \) hereafter) is constructed as,

\[
SKEW_t(\tau) = -\frac{W(t, \tau)}{V(t, \tau)^{3/2}}. \tag{16}
\]

To reconcile with our parametric framework, we derive the two moments in Eq. (15) under the data-generating process of the SVJ model. We relegate the complex derivations to Appendix B and briefly illustrate the role of crash risk on skewness here. Under the SVJ process delineated in Eqs. (1) and (2), the demeaned return is presented as follows:

\[
R(t, \tau) - E_t^Q [R(t, \tau)] = (\lambda (\mu^*) - 1/2) \int_t^{t+\tau} (V_u - E_0 [V_u]) \, du + \int_t^{t+\tau} \sqrt{V_u} \, dW_u^{(1)} + \int_t^{t+\tau} dJ_u - \lambda \mu^* Q \int_t^{t+\tau} E_0 [V_t] \, du.
\]

The terms \( \int_t^{t+\tau} (V_u - E_0 [V_u]) \, du \) and \( \int_t^{t+\tau} \sqrt{V_u} \, dW_u^{(1)} \) represent the future shocks of stochastic volatility and stock prices, respectively, and exhibit negative correlation, captured by a
negative $\rho$. Moreover, $\int_t^{t+\tau} dJ_u - \lambda \mu_J \int_t^{t+\tau} E_0 [V_t] du$ encompasses all the jump innovations in stock prices and has a third moment equal to:

$$\lambda \mu_J \int_t^{t+\tau} E_0 [V_t] du \left( \mu^2 + 3\sigma^2 \right).$$

With all else being equal, the jump size $\mu^*$ influences negative skewness in two ways. Firstly, a more negative $\mu^*$ heightens the negative covariance between stock price and volatility by the coefficient $\lambda(-\mu^*) - 1/2$. Secondly, given a direct relationship between $\mu^*$ and $\mu_J$, a more severe crash in $\mu^*$ suggests more negative third moments contributed by the jump process.

To further illustrate the impact of crash risk, we use our derivation in the Appendix B to present the skewness as a function of jump size $\mu^*$ and $\rho$ with all other parameters equal to our estimator and $V$ fixed to be the whole sample average. This exercise allows us to plot the skewness as a function of $\mu^*$, and $\rho$, respectively, in Figure 2. In Panel A of this figure, we plot the relation between skewness and $\mu^*$. Intuitively, as $\mu^*$ gets more negative, the distribution turns more negatively skewed. In addition, the relation reveals a concave function such that, when $\mu^*$ decreases, the skewness rapidly drops to a drastic level of around -7/-8. In comparison, the correlation also plays a monotonic role on skewness yet presents a more flat function curve and a lower level of negative skewness. Specifically, even when $\rho$ reaches -1, the skewness under our calibration is still at a moderate level of around -1.15.

Although the skewness measure is highly related to the jump size, it is also related to the level of stochastic volatility. Concurrently, the current level of $V_t$ steers both the second and third moments, and hence, the skewness. Drawing on these insights, we fashion our CIX as a distinct measure of crash risk, adjusting for the impact of $V_t$, rather than resorting to skewness, which is contingent on both crash and stochastic volatility risks.

4 Empirical Results

In this section, we present the results from our Stochastic Volatility-Jump (SVJ) model estimation, subsequently examining the Crash Index (CIX) and its potential implications.

4.1 Model Estimation

Panel A of Table 1 provides a detailed view of our parameter estimates. Significantly, $\mu^*$—the focal parameter of our research—stands at an annualized -16.16 percent, demonstrating statistical significance. Our results also highlight a substantial jump intensity $\lambda$ and a negative correlation between stochastic volatility and stock returns, represented by the $\rho$ parameter. Additionally, $k_v = 5.88$ signifies the mean-reversion coefficient under the physi-
cal measure, while the difference $k_v - \eta_v$ approximates to 2.6, serving as the mean-reversion coefficient under the risk-neutral measure. Accordingly, our findings concur with the established view that option-implied volatility mean-reversion is more sluggish. To provide further insight into our state-dependent jump model, we present the estimated conditional jump arrival intensity per year in Figure 1. The time series of this option-implied jump intensity harmonizes with the dynamics of option-implied volatility. Although the jump intensity $\lambda V_t$ is mean-reverting, it can reach extreme values during periods of market turbulence, and these extremes become more pronounced over time. The average jump arrival intensity per year is around 0.63.

In Panel B of Table 1, we encapsulate the model’s proficiency in fitting the joint moments of $(y_t, V_t)$. By normalizing all residuals of the moments in equation (9) using their respective standard deviations, we present the mean and T-statistics values. None of the joint moments of $(y_t, V_t)$ significantly deviate from zero. In summary, our estimates not only render robust economic interpretations, but they also resonate with the original estimator in Pan (2002). Despite our refined sampling and moment selection strategies, our estimation benefits from a longer sample and ensures efficiency.

4.2 Crash Risk Curve

With our model successfully estimated, we employ the estimator to infer a jump risk premium $\mu_t^*(\tau_i, K_i)$ for each option with maturity $\tau_i$ and strike price $K_i$, as defined in Eq. (12)

$$\mu_t^*(\tau_i, K_i) = g^\mu \left( f, V_t, \vartheta^\perp, r_t, q_t, \tau_i, \frac{K_i}{S_t} \right).$$

Instead of relying on option price data, we utilize the interpolated Black-Scholes implied volatility surface data from OptionMetrics to infer $\mu_t^*(\tau_i, K_i)$ at various grid points, illustrating the patterns in the crash risk surface. The use of the volatility surface is a standard practice to filter outliers and show how price patterns change with respect to strike prices and maturities. It also enables a direct comparison of our implied $\mu^*$ surface with the implied volatility surface.

In alignment with the VIX and our construction of CIX, we concentrate on the 30-day maturity data in the volatility surface. By fixing the maturity, we can represent a crash risk curve that depicts option-implied $\mu^*$ concerning strike prices. For each day $t$, $\mu_t^*(k)$ is presented as a function of normalized moneyness $k = K/S$. In Figure 3, the daily average of $\mu_t^*$ is plotted against moneyness $k$. Additionally, we juxtapose the $\mu^*$ curve with its analogous

\[\text{15} \text{We also investigate the CIX constructed from the crash surface. The constructed variable highly correlates with the CIX from price data yet is smoother.}\]
product, the daily average of the volatility curve. As well documented in the literature, the volatility curve exhibits a monotonic behavior in the range \( k \in [0.9, 1.0] \), consistent with the "volatility smirk" observation that out-of-the-money (OTM) puts are abnormally expensive relative to the Black-Scholes model.\(^{16}\) Contrarily, the crash risk curve reveals a U-shape in the short-term (30 days), where OTM options with moneyness around 0.95-0.98 imply a more negative jump size. This pattern coincides with the expectation that the most frequently traded OTM put options would be acutely sensitive to anticipated future crashes, leading us to construct CIX as an average of \(-\mu\), concentrating on strike prices within the most informative range. In comparison, we also plot the crash risk curve at 91-days to maturity. The longer-maturity curve turns into a monotonic, flat shape, which suggests a distinct role the crash risk plays in the short-run.

The divergence in the shapes of these two curves emphasizes the distinct roles crash risk and volatility play in clarifying the disparity between out-of-the-money (OTM) and at-the-money (ATM) options. We quantify this disparity by calculating the difference between implied volatilities at the OTM and ATM moneyness ranges. Specifically, we measure this discrepancy as:

\[
IV_{sprd} = \frac{1}{N_i} \sum_{s=1}^{N_i} \sigma_t(k_s) - \sigma_t(k = 1), \quad \text{s.t.} \, k_s \in [0.93, 0.97],
\]

where \( \sigma_t(k) \) represents the implied volatility on the 30-day volatility surface at each time \( t \). We employ this implied volatility spread (hereafter referred to as IVsprd) as a tool to gauge the difference between OTM and ATM option prices, converted into the volatility space for the purpose of unit normalization. We further associate this time series with the level of volatility represented by VIX and crash risk captured by CIX in the subsequent section.

4.3 Explaining the Dynamics of CIX

Given the observed patterns in the crash risk curve and our successfully estimated model, we construct the daily CIX as delineated in previous sections and examine its time-series dynamics. We depict the time-series of the CIX alongside the VIX and SKEW in Figure 4. To reduce noise, we apply an Exponential Weighted Moving Average (EWMA) to CIX, VIX, and other related variables. The EWMA of a time-series \( X_t \) is defined as

\[
EMA(X, \eta)_{t-1} = (1 - \eta) \sum_{\tau=0}^{t-1} \eta^{\tau} X_{t-\tau-1}.
\]

\(^{16}\)To study crash risk, we focus on the moneyness range for OTM put options, finding that the crash risk curve for in-the-money (ITM) puts scarcely provides information.
Contrary to what one might expect, and as our initial hypothesis suggested, the VIX and the jump risk premium encompassed in the CIX function as two distinct risk sources, showing minimal co-movement over time. Unlike traditional uncertainty measures, the CIX encapsulates investors’ anticipation of crash risk, a trend particularly noticeable during the Covid sample period. Moreover, the option implied SKEW reveals a strong co-movement with CIX. This co-movement further validates our estimation of jump size since \( \mu^* \) drives the major variation of the skewness, as derived under our SVJ model. Notably, both the moving averages of CIX and SKEW peaked in October and November 2017—a unique period that did not witness a crash in the stock market but rather experienced a rally of over five percent. Moreover, the VIX hit a historic low of 9.14 on 11/03. The fact that a pronounced divergence between OTM and ATM options captured in CIX occurs when VIX is extremely low highlights the difference between CIX and VIX in driving asset pricing patterns.

To elucidate this distinction and gain a deeper understanding of the dynamics of related variables, we present summary statistics for CIX, VIX, the skewness index (SK), and implied volatility spread (IVsprd) in Panel A of Table 2. This includes the mean, standard deviation (Std), and correlations for CIX, VIX, SK, IVsprd, and the index return over the entire sample period. Remarkably, our CIX measure is less persistent than VIX, with an auto-regressive coefficient of 0.71, and shows only a weak correlation with both the volatility index and index returns. Furthermore, the option-implied skewness, a parallel measure of the crash risk impact, is influenced by both the CIX and VIX indexes. The correlation between SK and VIX reveals the advantage of CIX as a more refined measure of crash risk, controlling for the volatility risk in our parametric framework. In line with expectations, both the VIX and CIX correlate with the IVsprd, signifying two components that drive the disparity between OTM and ATM options. On the one hand, CIX positively explains the spread since a higher jump size \( \mu^* \) signifies a greater difference between OTM and ATM options. On the other hand, a higher level of volatility tends to flatten the volatility surface, as OTM options exhibit lower sensitivity to volatility (Vega). This pattern clarifies the negative relationship between IVsprd and VIX.

To delve deeper into these differences, we analyze the relationship between the innovations of CIX, VIX, SK, and IVsprd. We define the innovations of these variables as,

\[
\Delta X_t = X_t - \text{EMA}(X, \eta = 0.7)_{t-1}, \quad X = \{\text{CIX, VIX, SK, IV sprd}\}.
\]

This definition allows us to isolate unexpected shocks in the time-series by accounting for the past values’ exponential moving average, with a slow decay rate of \( \eta = 0.7 \) to smooth out noise, considering our CIX measure’s lower persistence.\(^{17}\) As presented in Panel B of
Table 2, the innovations in CIX exhibit a weak correlation with VIX and the index return. Similarly, the innovations in SK and IVsprd correlate with both $\Delta CIX$ and $\Delta VIX$.

Further, in Panel A of Table 3, we investigate the relationship between a related variable $X_t$ and the dynamics of CIX by executing the following regression:

$$\Delta CIX_t = \text{constant} + b \Delta X_t + c 1_{\Delta X_t > Q(\Delta X_t, \alpha = 0.9)},$$

(18)

where $Q(\Delta X_t, \alpha = 0.9)$ is the 90% quantile of the innovations of our explaining variable $X_t$. The dummy variable $1_{\Delta X_t > Q(\Delta X_t, \alpha = 0.9)}$ targets tail events when the increment of $X_t$ surpasses its 90% quantile, allowing the regression to correlate the changes in CIX with other variables and highlight the effect of extreme alterations in dynamics.

Before we move to the our regression analysis, it is worthy noting that all the standard errors throughout our paper employs the Newey-West estimator to account for serial correlation and heteroskedasticity. In accord with the correlation matrix results, both option-implied skewness and the divergence between OTM and ATM implied volatility significantly explain the dynamics of CIX. This importance also manifests during tail events when SKEW and IVsprd surge. Moreover, VIX remains uncorrelated with CIX, emphasizing CIX as a distinct risk from volatility. Utilizing an alternative signaling variable, we compute the daily ratio of put option to call option volume (P/C), and find that this volume-based ratio positively influences CIX, inferring that trading volume contains information about both risk resources.

Beyond the risk measures derived from option data, we probe the association between $\Delta CIX_t$ and various macroeconomic variables to detect any intersection between jump risk and other risk sources. We examine the bond price noise measure (Noise) in Hu et al. (2013), the term spread (calculated by the difference between 10-year and 2-year treasury yields (Term)), the TED spread (calculated by the difference between the three-month Treasury bill rate and the three-month LIBOR based in U.S. dollars), and the default spread (calculated by the yield disparity between the AAA and BAA corporate bond index (Dsprd)). None of these factors significantly influence the dynamics of our jump risk measure, reinforcing crash risk as a distinctive and independent asset pricing element.

Interestingly, there is a tenuous connection between the bond price noise measure and CIX. Even though this measure originates from bond price and thus does not relate to CIX based on the SPX index, this observation prompts us to examine the influence of liquidity on the dynamics of CIX. From this standpoint, in Panel B of Table 3, we adjust for $1_{3\text{rd} \text{ Friday}}$, the dummy variable for the third Friday of each month when typical options expire. When factoring in the expiration day of the predominantly traded options, CIX is significantly elevated, signifying illiquidity components within CIX. Additionally, we include a dummy
variable for the FOMC announcement days, but find no significant difference in CIX on days with or without an announcement.

4.4 Reconciling with Option-Implied Skewness

Supported by our theoretical derivations and observable patterns in the data, we establish a robust correlation between CIX and skewness, despite their distinct emphases and empirical properties. Specifically, SKEW aims to estimate the moments of the risk-neutral density and, therefore, relies on both separate risk sources examined in this paper: VIX and CIX. Unlike the non-parametric approach, which depends on the entire collection of options to gauge skewness, our methodology enables the estimation of the crash parameter for each individual option, allowing a focus on those contracts that are informationally richer.

Additionally, as outlined in Table 2, our measure of crash risk (CIX) exhibits more time-series fluctuations and more extreme values compared to SKEW. This result aligns with our perspective that CIX is an improved measure of crash risk, designed to capture the tail behavior of asset prices.

In Figure 5, we juxtapose the non-parametric measure of skewness with the skewness implied by our model under various parameter configurations. Specifically, we investigate the model-implied skewness driven by stochastic volatility \( V_t \), maintaining constant parameters, consistent with our estimation using ATM options. The observed co-movement between the non-parametric and model-implied measures corroborates our SVJ model estimation.

We also explore a variant of model-implied skewness by setting \( \rho = 0 \), and as anticipated, this altered time-series closely resembles the original model-implied skewness, reflecting the limited impact of \( \rho \) on skewness. Moreover, the red dashed line in the figure portrays skewness using time-varying CIX extracted from OTM options, in contrast to the \( \mu^* \) estimated from ATM options.

Notably, the skewness implied by time-varying CIX demonstrates significant fluctuations and a more pronounced magnitude. This observation is congruent with the volatile dynamics and valuable information extracted from OTM options to form CIX. From this standpoint, CIX serves as a superior crash risk measure, emphasizing the tail events of asset pricing dynamics rather than simply revealing attributes of a stable risk-neutral distribution.

4.5 Time-Series Predictability

Given the capability of CIX to reflect the tail crash risk, we further highlight its asset pricing implications by investigating the stock market’s response to significant changes in CIX. In Table 4, we monitor market performance following extreme fluctuations in CIX, forging a
path towards a more nuanced comprehension of market dynamics and crash risk.

Analogous to the tail-event dummy variable employed in Table 3, we isolate extreme events by concentrating on the days with the highest percentiles of $\Delta CIX$, and compute the average stock index return on the subsequent day. Panel A of Table 4 shows that a sharp 2% increase in CIX foretells a marked 47.4 basis point decrease in the index return, a stark contrast to the overall sample average of 3-4 basis points. As we examine tail events in the top 5, 10, 15, and 20 percentiles, the extent of the decline following a CIX increase diminishes but remains substantial. Moreover, as the selected percentile broadens to 30, the significance of the negative return spread wanes.

In a contrasting vein, Panel A of Table 4 aligns our examination with findings for VIX as described in Hu et al. (2022). The results unveil that a significant rise in VIX typically signals positive returns for the S&P 500 index. Consistently, we note that a 2% hike in VIX is succeeded by a robust 59.88 basis point enhancement in stock market return. This upward trend is maintained even when the cutoff percentage is reduced. This disparity between CIX and VIX in relation to subsequent stock market performance emphasizes the importance of distinguishing jump risk from stochastic volatility risk, a pivotal theme explored in our paper.

Furthermore, we replicate this analysis for option-implied skewness and implied volatility spread. The spikes in SKEW similarly foreshadow a noteworthy negative index return, albeit at a reduced magnitude. Contrarily, increases in IVsprd do not correlate with significant next-day returns. These observations resonate with our thesis that both SKEW and IVsprd are influenced by crash and volatility risks, thereby conveying limited insights into potential future market crashes.

Moreover, to glean a deeper understanding of the informational value of our option-implied jump risk measure, we investigate the stock market’s response to a pronounced decline in the CIX, as presented in Panel B of Table 4. We examine the tail events where the CIX decrease aligns with the same percentage probabilities in our sample. Remarkably, when the CIX experiences its most drastic 2% drop, the subsequent day’s index return averages a positive 20.28 basis points. Although this positive effect isn’t as potent as the negative return triggered by an escalating fear of a crash, as shown in our previous test, it still establishes a noteworthy pattern. Interestingly, for VIX, we discover that all significant reductions fail to forecast any significant behavior in index returns. This insight helps to differentiate jump risk from volatility risk, the latter showing an asymmetric behavior: a high VIX typically portends a market downturn, but this inverse relationship softens when the VIX is low.

For comparison, we repeat the same exercise for the unanticipated changes in SKEW and IVsprd. In the case of a sudden drop in SK and IVsprd, as reported in Panel B, both variables
precede a significant positive return on the following day. This outcome relates to their negative dependence on VIX; a sudden decrease in SK/IVsprd might occur simultaneously with a sudden increase in VIX, thus leading to positive returns on the next day. In sum, the extreme shifts in skewness and implied volatility spread reveal less information about future market crashes. This observation underscores our argument that these two variables depend on both CIX and VIX, highlighting the necessity to develop the CIX as an isolated crash risk indicator.

Our empirical findings consistently demonstrate a strong inverse relationship between $\Delta CIX$ and subsequent daily stock market returns. We extend our analysis to examine whether this negative correlation between CIX and future stock returns remains unaffected by spikes in other variables, especially the option-implied moments embodied in VIX and SKEW. Utilizing the same tail-event dummy variable,

$$1_{\Delta X_t} = 1_{\Delta X_t > Q(\Delta X_t;0.9)},$$

where $X = CIX, VIX, SKEW$, we incorporate them into our predictive model for the next-day index return. This approach enables us to assess the effects of tail events when CIX surges and to account for possible interactions with VIX and SKEW. For example, besides the tail dummy of CIX itself, $1_{\Delta CIX_t}$, we include $1_{\Delta VIX_t}$ and the product term $1_{\Delta CIX_t}1_{\Delta VIX_t}$ to control for simultaneous extreme changes in both CIX and VIX.

In Panel A of Table 5, we display the results of this regression employing tail dummy variables. Consistent with earlier findings, a single dummy regression using CIX or SKEW forecasts a negative return, while using VIX anticipates a positive return. Importantly, the interaction terms with VIX do not significantly influence subsequent returns, highlighting the separate nature of risk surges in CIX and VIX. Moreover, factoring in interactions with SKEW does not diminish the predictive strength of CIX. Indeed, considering both tail events enhances the predictive potency of CIX as an early indicator of impending market declines.

In summary, our evidence underlines that a surge in CIX symbolizes an increase in forward-looking crash risk as perceived by investors, thereby presaging negative future stock returns. We clarify this conclusion through the following predictive regressions:

$$\text{Ret}_{t+1} = \text{constant} + b_1 CIX_t + b_2 1_{\Delta CIX_t} + \text{controls}.$$  

Here, the regular terms with coefficients $b_1$ utilize CIX as a predictor, while the tail-dummy with coefficients $b_2$ captures the effect at tail events when CIX shows a significant rise.

In Panel B of Table 5, we discuss the results employing $CIX_t$ as a forecaster of the next-day index returns. The innovations of CIX itself negatively influence subsequent returns. Further, the inclusion of the tail-dummy variable confirms significant predictability.
for both the predictor and the tail-events. Additionally, introducing SKEW and VIX as control variables does not diminish the tail-predictive power, underscoring the crucial impact of an unexpected surge in crash risk on stock returns. Lastly, to better understand the illiquidity component of CIX, we include a dummy variable for the third Friday of each month as a robustness check. Interestingly, while CIX levels rise on these option-expiring dates, its predictive value remains unchanged after accounting for this expiration factor. This observation implies that surges in CIX carry substantial information about future crash risk, rather than merely reflecting mispricing of OTM options due to liquidity constraints.

In conclusion, our empirical research highlights a potent inverse correlation between the Crash Index (CIX) and future stock market returns. Importantly, this compelling relationship remains robust irrespective of the shifts in uncertainty as captured by the VIX index nor the non-parametric measure of skewness. This pronounced link is particularly evident during instances of abrupt and extreme CIX fluctuations, underscoring the profound impact of elevated forward-looking crash risk as a significant short-term determinant in asset pricing dynamics.

5 Conclusions

In this paper, we have pioneered a methodology to gauge forward-looking crash risk as implied from option prices. Utilizing the tractable SVJ model, this parametric approach isolates the jump size component from the stochastic volatility encapsulated within uncertainty risk. Our method extends beyond the traditional Black-Scholes model, paralleling the construction of the implied volatility surface and facilitating the creation of an option-implied crash-risk curve. This framework uniquely empowers us to extract crash risk insights from OTM options while simultaneously controlling for latent state variables.

Our method’s efficacy is underscored by its strong correlation with non-parametric option-implied skewness. Nevertheless, we have crafted our CIX as a nuanced measure of crash risk, designed to adjust for the influence of $V_t$, and illuminate the tail risk aspects of asset pricing dynamics. In juxtaposition, option-implied skewness is reliant on both crash and stochastic volatility risks and epitomizes the more smooth characteristics of the risk-neutral density.

Empirically, we unearth a persistent negative relationship between spikes in the CIX index and subsequent stock market index returns. This association endures even after accounting for other option-implied risk measures and underscores the valuable insights that OTM put options offer regarding impending market crashes.

Looking ahead, our innovative method for implying crash risk offers fertile ground for future research. Extensions beyond the SVJ model can pave the way to uncover additional risk
resources that influence option price deviations. Such exploration could lead to a more comprehensive examination of the corresponding asset pricing implications, further enhancing our understanding of market dynamics and risks.
References


6 Tables and Figures

Figure 1. Model estimated conditional jump intensity, measured in number of times per year. The conditional jump intensity is $\lambda V_t$ per year. We use the IS-GMM approach specified in Eq. (9), and extract the stochastic volatility $V_t$ from a near to at-the-money call option with around 30 days to maturity as in Eq. (8). The dashed read line plots the average arrival intensity, which is around 0.63 per year.
Figure 2. Risk-Neutral Skewness as a Function of the Jump Size and Price-Volatility Correlation. We use our SVJ model specified in Eqs. (3) and (4) to compute the risk-neutral moments and skewness defines in Eqs. (15) and (16). We use our derivation in the Appendix B to present the skewness as a function of jump size \( \mu^* \) and \( \rho \) with all else being equal and \( V \) equals the whole sample average. \( \rho \) is the correlation between Brownian motions in stock price and stochastic volatility dynamics.
Figure 3. BS Implied Volatility and SVJ Model Implied Jump Size Plotted against Strike-to-Spot Ratio. Using the volatility surface data, we plot the BS implied volatility $\sigma^I$ and SVJ model implied $\mu^I$ specified in Eq. (12) against the strike-to-spot ratio $k = K/S$. The above panel plots the curve with 30 days to maturity and the below plots the curve with 91 days to maturity.
To reduce noise, we apply an Exponential Weighted Moving Average (EWMA) to CIX, VIX, and SKEW. The EWMA of a time-series $X_t$ is defined as $EMA(X, \eta)_{t-1} = (1 - \eta) \sum_{\tau=0}^{t-1} \eta^\tau X_{t-\tau-1}$. We set $\eta = 0.97$ in this plot. The shaded areas are NBER recession periods.

Figure 4. Time Series of CIX, VIX, and SKEW.
Figure 5. Non-parametric and Model-implied Skewness. We illustrate the non-parametric measure of skewness, as defined in Eq. (16), by plotting it alongside the skewness implied by our model under different parameter sets. We explore the model-implied skewness by keeping all parameters constant, as in our estimation, and basing it on $V_t$. Additionally, we introduce a variation by setting the correlation between stock price and volatility, denoted as $\rho$, to zero. Finally, rather than employing the $\mu^*$ estimated from ATM options, the red dashed line in the figure represents the skewness using the time-varying CIX, which is extracted from OTM options.
Table 1. Model Estimation Results

Panel A: SVJ estimation using 1996-2021 daily price data

<table>
<thead>
<tr>
<th></th>
<th>$k_v$</th>
<th>$\bar{v}$</th>
<th>$\sigma_v$</th>
<th>$\rho$</th>
<th>$\lambda$</th>
<th>$\mu$</th>
<th>$\sigma_J$</th>
<th>$\eta_s$</th>
<th>$\eta_v$</th>
<th>$\mu^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Est</td>
<td>5.88</td>
<td>0.0108</td>
<td>0.29</td>
<td>-0.52</td>
<td>23.9</td>
<td>-1.00%</td>
<td>2.97%</td>
<td>3.10</td>
<td>3.19</td>
<td>-16.16%</td>
</tr>
<tr>
<td>T-stat</td>
<td>[45.12]***</td>
<td>[8.70]***</td>
<td>[15.54]***</td>
<td>[-15.11]***</td>
<td>[15.71]***</td>
<td>[-0.14]</td>
<td>[5.83]***</td>
<td>[16.65]***</td>
<td>[-37.32]***</td>
<td></td>
</tr>
</tbody>
</table>

Panel B: SVJ fitting the joint moments of $y_t$ and $V_t$

<table>
<thead>
<tr>
<th></th>
<th>$y_t$</th>
<th>$y_t^2$</th>
<th>$y_t^3$</th>
<th>$y_t^4$</th>
<th>$V_t$</th>
<th>$V_t^2$</th>
<th>$y_tV_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>20.14</td>
<td>0.35</td>
<td>1.15</td>
<td>-0.20</td>
<td>0.03</td>
<td>0.01</td>
<td>1.10</td>
</tr>
<tr>
<td>Tstats</td>
<td>[0.26]</td>
<td>[0.09]</td>
<td>[0.21]</td>
<td>[-0.06]</td>
<td>[0.04]</td>
<td>[0.02]</td>
<td>[0.20]</td>
</tr>
</tbody>
</table>

*, **, and *** represent significance of the two-tail test at the 10%, 5%, and 1% level, respectively.

We report the estimates of the parameters $\vartheta = (\kappa_v, \bar{v}, \sigma_v, \rho, \lambda, \mu, \sigma_J, \eta_s, \eta_v, \mu^*)$ using the IS-GMM approach. In addition, we report the average fitting error and T-stats of moments conditions. The fitting error $\epsilon$ of each moments condition is specified as follows,

$$
\begin{align*}
\epsilon_{y1}^y &= y_t - M_1 (V_{t-1}, \vartheta) , \\
\epsilon_{y2}^y &= y_t^2 - M_2 (V_{t-1}, \vartheta) , \\
\epsilon_{y3}^y &= y_t^3 - M_3 (V_{t-1}, \vartheta) , \\
\epsilon_{y4}^y &= y_t^4 - M_4 (V_{t-1}, \vartheta) , \\
\epsilon_{v1}^y &= V_t - M_5 (V_{t-1}, \vartheta) , \\
\epsilon_{v2}^y &= V_t^2 - M_6 (V_{t-1}, \vartheta) , \\
\epsilon_{yv} &= y_tV_t - M_7 (V_{t-1}, \vartheta) .
\end{align*}
$$
Table 2. Summary Statistics

Panel A: Summary Statistics, Level

<table>
<thead>
<tr>
<th>Summary Statistics</th>
<th>Correlation Matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
</tr>
<tr>
<td>CIX</td>
<td>15.50</td>
</tr>
<tr>
<td>VIX</td>
<td>20.29</td>
</tr>
<tr>
<td>IVsprd</td>
<td>4.70</td>
</tr>
<tr>
<td>SKEW</td>
<td>1.24</td>
</tr>
<tr>
<td>Ret</td>
<td>9.78</td>
</tr>
</tbody>
</table>

Panel B: Summary Statistics, Innovation

<table>
<thead>
<tr>
<th>Summary Statistics</th>
<th>Correlation Matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
</tr>
<tr>
<td>ΔCIX</td>
<td>0.00</td>
</tr>
<tr>
<td>ΔVIX</td>
<td>0.00</td>
</tr>
<tr>
<td>ΔIVsprd</td>
<td>0.00</td>
</tr>
<tr>
<td>ΔSKEW</td>
<td>0.00</td>
</tr>
<tr>
<td>Ret</td>
<td>9.78</td>
</tr>
</tbody>
</table>

In Panel A, we report the mean, standard deviation (Std), and correlations for CIX, VIX, SK, IVsprd, and the index return over the entire sample period. We also report the correlation among these variables. In Panel B, we analyze the relationship between the innovations of CIX, VIX, SK, and IVsprd. We define the innovations of these variables as $ΔX_t = X_t - EMA(X, \eta = 0.7)t_{t−1}$ for $X = \{CIX, VIX, SK, IVsprd\}$ respectively.
Table 3. How Related Variables Explain the Dynamic of CIX

<table>
<thead>
<tr>
<th>Panel A: $\Delta CIX_t = \text{constant} + b \Delta X_t + c \text{1}_{\Delta X_t &gt; Q(\Delta X_t, \alpha = 0.9)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_t = IV\ sprd$</td>
</tr>
<tr>
<td>$X_t$</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>$1_{X_t &gt; Q(\Delta X_t, \alpha = 0.9)}$</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>$R^2$ (%)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B: $\Delta CIX_t = \text{constant} + b \Delta X_t + c \text{1}_{\Delta X_t &gt; Q(\Delta X_t, \alpha = 0.9)} + \text{Other Dummies}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_t = IV\ sprd$</td>
</tr>
<tr>
<td>$X_t$</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>$1_{X_t &gt; Q(\Delta X_t, \alpha = 0.9)}$</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>$1_{3rd\ Friday}$</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>$1_{FOMC}$</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>$R^2$ (%)</td>
</tr>
</tbody>
</table>

$Q(\Delta X_t, \alpha = 0.9)$ is the 90% quantile of the innovations of our explaining variable $X_t$. The dummy variable $1_{\Delta X_t > Q(\Delta X_t, \alpha = 0.9)}$ identifies the tail events when the increment of $X_t$ exceeds its 90% quantile. The SKEW and implied volatility spread are defined in Eqs. (16), (17). We calculate the daily ratio of put option to call option volume (P/C). We include the bond price noise measure (Noise) in Hu et al. (2013), the term spread (measured by the difference between the 10-year and 2-year treasury yield (Term)), the TED spread (measured by the difference between the three-month Treasury bill rate and the three-month LIBOR based in U.S. dollars), and the default spread (measured by the yield difference between the AAA and BAA corporate bond index (Dsprd)). $1_{3rd\ Friday}$ is the dummy variable for the third Friday of each month when typical options expires. $1_{FOMC}$ is the dummy variable for the FOMC announcement days. Reported in the squared brackets are the t-stats calculated using Newey-West standard errors.
Table 4. Average Next Day SPX Returns (bps) After Tail Events

Panel A: Avg Next-Day SPX Returns (bps) after Risk Measure Surges

<table>
<thead>
<tr>
<th>Prob(Δy ≥ x)</th>
<th>y=CIX</th>
<th>y=VIX</th>
<th>y=SKEW</th>
<th>y=IV Sprd</th>
</tr>
</thead>
<tbody>
<tr>
<td>2%</td>
<td>-47.41</td>
<td>59.88</td>
<td>-29.08</td>
<td>-23.87</td>
</tr>
<tr>
<td></td>
<td>[-2.89]**</td>
<td>[2.03]*</td>
<td>[-2.03]**</td>
<td>[-1.47]</td>
</tr>
<tr>
<td>5%</td>
<td>-28.96</td>
<td>29.73</td>
<td>-24.84</td>
<td>-12.63</td>
</tr>
<tr>
<td></td>
<td>[-3.39]**</td>
<td>[2.14]*</td>
<td>[-3.27]***</td>
<td>[-1.42]</td>
</tr>
<tr>
<td>10%</td>
<td>-18.29</td>
<td>17.52</td>
<td>-13.31</td>
<td>-9.76</td>
</tr>
<tr>
<td></td>
<td>[-3.44]**</td>
<td>[2.12]*</td>
<td>[-2.56]***</td>
<td>[-1.66]</td>
</tr>
<tr>
<td>15%</td>
<td>-12.16</td>
<td>14.04</td>
<td>-12.23</td>
<td>-6.53</td>
</tr>
<tr>
<td></td>
<td>[-2.89]**</td>
<td>[2.36]**</td>
<td>[-3.10]***</td>
<td>[-1.46]</td>
</tr>
<tr>
<td>20%</td>
<td>-10.41</td>
<td>12.06</td>
<td>-10.42</td>
<td>-4.77</td>
</tr>
<tr>
<td></td>
<td>[-2.90]**</td>
<td>[2.52]**</td>
<td>[-3.09]***</td>
<td>[-1.26]</td>
</tr>
<tr>
<td>30%</td>
<td>-4.65</td>
<td>8.08</td>
<td>-7.34</td>
<td>-3.71</td>
</tr>
<tr>
<td></td>
<td>[-1.61]</td>
<td>[2.29]**</td>
<td>[-2.74]***</td>
<td>[-1.28]</td>
</tr>
</tbody>
</table>

Panel B: Avg Next-Day SPX Returns (bps) after Risk Measure Drops

<table>
<thead>
<tr>
<th>Prob(Δy ≥ x)</th>
<th>y=CIX</th>
<th>y=VIX</th>
<th>y=SKEW</th>
<th>y=IV Sprd</th>
</tr>
</thead>
<tbody>
<tr>
<td>2%</td>
<td>20.28</td>
<td>-4.99</td>
<td>25.22</td>
<td>35.70</td>
</tr>
<tr>
<td></td>
<td>[1.70]</td>
<td>[-0.26]</td>
<td>[1.46]</td>
<td>[1.92]</td>
</tr>
<tr>
<td>5%</td>
<td>19.55</td>
<td>-3.39</td>
<td>21.32</td>
<td>30.17</td>
</tr>
<tr>
<td></td>
<td>[2.47]**</td>
<td>[-0.34]</td>
<td>[2.33]**</td>
<td>[2.99]***</td>
</tr>
<tr>
<td>10%</td>
<td>11.60</td>
<td>4.25</td>
<td>22.08</td>
<td>21.59</td>
</tr>
<tr>
<td></td>
<td>[2.24]**</td>
<td>[0.71]</td>
<td>[3.93]***</td>
<td>[3.52]***</td>
</tr>
<tr>
<td>15%</td>
<td>8.32</td>
<td>3.72</td>
<td>19.13</td>
<td>18.99</td>
</tr>
<tr>
<td></td>
<td>[2.00]**</td>
<td>[0.83]</td>
<td>[4.49]***</td>
<td>[4.01]***</td>
</tr>
<tr>
<td>20%</td>
<td>6.64</td>
<td>2.97</td>
<td>15.77</td>
<td>17.29</td>
</tr>
<tr>
<td></td>
<td>[1.92]</td>
<td>[0.82]</td>
<td>[4.40]***</td>
<td>[4.45]***</td>
</tr>
<tr>
<td>30%</td>
<td>8.18</td>
<td>2.58</td>
<td>13.37</td>
<td>11.85</td>
</tr>
<tr>
<td></td>
<td>[2.94]***</td>
<td>[0.97]</td>
<td>[4.74]***</td>
<td>[3.99]***</td>
</tr>
</tbody>
</table>

We report the average next day SP&500 index returns after surges of related variables $y = CIX, VIX, SKEW, IV sprd$. We focus on various levels of tail probability such that Prob(Δy ≥ x) equals 2%, 5%, 10%, 15%, 20%, 30%, respectively. Reported in the squared brackets are the t-stats of the average returns.
Table 5. Predict SPX Returns

**Panel A: Predict SPX Returns with Tail-Dummy Variables**

\[ 1_{\Delta X_t = 1_{\Delta X_t > Q(\Delta X_t, 0.9)}} X = CIX, VIX, SKEW \]

<table>
<thead>
<tr>
<th></th>
<th>( \Delta CIX_t )</th>
<th>( \Delta VIX_t )</th>
<th>( \Delta SKEW_t )</th>
<th>( 1_{\Delta CIX_t} * 1_{\Delta VIX_t} )</th>
<th>( 1_{\Delta CIX_t} * 1_{\Delta SKEW_t} )</th>
<th>( R^2(%) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Ret_{t+1} )</td>
<td>-0.25</td>
<td>0.15</td>
<td>-0.19</td>
<td>-0.37</td>
<td>0.09</td>
<td>0.37</td>
</tr>
<tr>
<td></td>
<td>[-4.39]***</td>
<td>[2.32]**</td>
<td>[-3.71]***</td>
<td>[-1.51]</td>
<td>[0.68]</td>
<td></td>
</tr>
<tr>
<td>( Ret_{t+1} )</td>
<td>-0.20</td>
<td>0.23</td>
<td>-0.13</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>[-4.27]***</td>
<td>[3.08]**</td>
<td>[-2.01]*</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

We use \( 1_{\Delta X_t = 1_{\Delta X_t > Q(\Delta X_t, 0.9)}} \) as tail-dummy variable for CIX, VIX, and SKEW to predict the next day returns \( Ret_{t+1} \). Here, \( Q(\Delta X_t, 0.9) \) is the 90% quantile for the innovations of each variable. Reported in the squared brackets are the t-stats calculated using Newey-West standard errors.

---

**Panel B: Predict SPX Returns with \( \Delta CIX \) and Tail-Dummy Variables**

<table>
<thead>
<tr>
<th></th>
<th>( \Delta CIX_t )</th>
<th>( \Delta VIX_t )</th>
<th>( \Delta SKEW_t )</th>
<th>( 1_{3rd Friday} )</th>
<th>( R^2(%) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Ret_{t+1} )</td>
<td>-3.30</td>
<td>-0.15</td>
<td>-0.54</td>
<td>-0.01</td>
<td>0.40</td>
</tr>
<tr>
<td></td>
<td>[-4.20]***</td>
<td>[-2.34]**</td>
<td>[-2.74]***</td>
<td>[-0.08]</td>
<td></td>
</tr>
<tr>
<td>( Ret_{t+1} )</td>
<td>-2.16</td>
<td>-0.17</td>
<td>-0.54</td>
<td></td>
<td>0.48</td>
</tr>
<tr>
<td></td>
<td>[-2.44]**</td>
<td>[-2.62]**</td>
<td>[-2.74]***</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( Ret_{t+1} )</td>
<td>-2.26</td>
<td>-0.16</td>
<td>-0.54</td>
<td></td>
<td>1.23</td>
</tr>
<tr>
<td></td>
<td>[-2.58]**</td>
<td>[-2.46]**</td>
<td>[-2.74]***</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( Ret_{t+1} )</td>
<td>-0.69</td>
<td>4.63</td>
<td>4.63</td>
<td></td>
<td>1.36</td>
</tr>
<tr>
<td></td>
<td>[-0.68]</td>
<td>[3.46]**</td>
<td>[3.45]**</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\( R^2(\%) \) is calculated using Newey-West standard errors.
Appendices

Appendix A  Option pricing

This appendix provides option pricing under the risk-neutral dynamics specified in Eqs. (3) and (4). Our derivation is a simplified version of the Appendix B in Pan (2002), where we take the interest rate $r$ and dividend yield $q$ as constant in the option pricing formula yet we update the $r$ and $q$ as if it is a constant per day in our empirical specification.

For any $c \in \mathbb{C}$, the time- $t$ conditional transform of $\ln S_T$, when well defined, is given by

$$
\psi^{\vartheta} (c, V_t, r, q, \tau) = \exp (-r\tau) \mathbb{E}_t^Q \left[ -e^{\ln S_T} \right].
$$

Under certain integrability conditions (Duffie et al. (2000)),

$$
\psi^{\vartheta} (c, V_t, r, q, \tau) = \exp \left( \alpha_v (c, t, \vartheta) + \beta_v (c, t, \vartheta) v + \beta_r r + \beta_q q \right), \quad (A.1)
$$

where the coefficients in Eq. (B.1) $\alpha_v$, $\beta_r$, $\beta_q$, and $\beta_v$ are defined by

$$
\beta_v (c, t, \vartheta) = -\frac{a (1 - \exp (-\gamma_v t))}{2\gamma_v - (\gamma_v + b) (1 - \exp (-\gamma_v t))},
$$

$$
\alpha_v (c, t, \vartheta) = -\frac{\kappa_v^* \bar{\nu}^*}{\sigma_v^2} \left( (\gamma_v + b) \tau + 2 \ln \left[ 1 - \frac{\gamma_v + b}{2\gamma_v} (1 - e^{-\gamma_v \tau}) \right] \right)
$$

$$
\beta_r = (c - 1) \tau
$$

$$
\beta_q = -c * \tau,
$$

where $b = \sigma_c \rho c - \kappa_v^*$, $a = c(1 - c) - 2\lambda \left[ \exp (c \mu_j^* + c^2 \sigma_j^2/2) - 1 - c \mu^* \right]$, and $\gamma_v = \sqrt{b^2 + 4a\sigma_v^2}$.

The parameters superscripted by * denote the risk-neutral counterparts of those under the data-generating measure $P$. For example, $\kappa_v^* = \kappa_v - \eta^*$ and $\bar{\nu}^* = \kappa_v \bar{\nu}/\kappa_v^*$ are the risk-neutral mean-reversion rate and long-term mean, respectively, and $\mu_j^* = \ln (1 + \mu^*) - \sigma_j^2/2$ is the risk-neutral counterpart of $\mu_J$. While the square root and logarithm of a complex number $z$ are not uniquely defined, for notational simplicity the results are presented as if we are dealing with real numbers. To be more specific, we define, $\sqrt{z} = |z|^{1/2} \exp(i \arg(z)/2)$ and $\ln(z) = \ln |z| + i \arg(z)$, where for any $z \in \mathbb{C}$, $\arg(z)$ is defined such that $z = |z| \exp(i \arg(z))$, with $-\pi < \arg(z) \leq \pi$.

Letting $k_t = K_t/S_t$ be the time- $t$ "strike-to-spot" ratio, the time- $t$ price of a European-style call option with time-to-expiration $\tau_t$ can be calculated as

$$
C_t = S_t f (V_t, \vartheta, r, q, \tau_t, k_t)
$$

where $f : \mathbb{R}_+ \times \Theta \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow [0, 1]$ is defined by
\[ f(v, \vartheta, r, q, \tau, k) = P_1 - kP_2 \]

with

\[
P_1 = \frac{\psi(1, v, r, q, \tau)}{2} - \frac{1}{\pi} \int_0^\infty \frac{\text{Im}\left(\psi(1 - iu, v, r, q, \tau)e^{iu(\ln k)}\right)}{u} \, du
\]

\[
P_2 = \frac{\psi(0, v, r, q, \tau)}{2} - \frac{1}{\pi} \int_0^\infty \frac{\text{Im}\left(\psi(-iu, v, r, q, \tau)e^{iu(\ln k)}\right)}{u} \, du
\]

where \( \text{Im}(\cdot) \) denotes the imaginary component of a complex number.

It should be noted that, whenever applicable, all of expectations and probability calculations in this appendix are taken with respect to the riskneutral measure \( Q \). Further, with the simplified assumption that \( r \) and \( q \) are constant,

\[
\psi(1, v, r, q, \tau) = \exp(-q\tau), \quad \psi(0, v, r, q, \tau) = \exp(-r\tau)
\]

We calculate \( P_1 = \psi(1)\tilde{P}_1 \) and define \( \tilde{P}_1 \) as a real probability that can be calculated through the standard Lévy inversion formula to match compute an "in the money probability" using the characteristic function,

\[
\tilde{P}_1 = P\left(\tilde{X}_1 \leq \bar{x}\right) = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \frac{\text{Im}\left(\tilde{\psi}_1(u) \exp(-iu\bar{x})\right)}{u} \, du.
\]

where \( \bar{x} = (r - q)\tau - \ln k \), and where the random variable \( \tilde{X}_1 \) is uniquely defined by its characteristic function \( \tilde{\psi}_1(u) \) via

\[
\tilde{\psi}_1(u) = \frac{\psi(1 - iu) \exp(iu(r - q)\tau)}{\psi(1)}.
\]

The calculation for \( P_2 \) is done similarly by presenting \( P_2 = \psi(0)\tilde{P}_2 \), and defining

\[
\tilde{P}_2 = P\left(\tilde{X}_2 \leq \bar{x}\right) = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \frac{\text{Im}\left(\tilde{\psi}_2(u) \exp(-iu\bar{x})\right)}{u} \, du.
\]

where \( \bar{x} = (r - q)\tau - \ln k \), and where the random variable \( \tilde{X}_2 \) is uniquely defined by its characteristic function \( \tilde{\psi}_2(u) \) via

\[
\tilde{\psi}_2(u) = \frac{\psi(-iu) \exp(iu(r - q)\tau)}{\psi(0)}.
\]
Appendix B  Relation to Non-parametric Moments

For simplicity, we assume the dividend yield \( q = 0 \), and the short rate \( r \) is a constant in this section. For our purpose, we focus our calculation in Q measure and leave out the Q notation. The stock price \( S_t \) satisfies

\[
dS_t = S_trdt + S_t\sqrt{V_t}dB_{1,t}(Q) + dZ^Q_t - \mu^*S_t\lambda V_tdt
\]

The jump process \( dZ^Q_t \) is defined as in our model, with jump intensity \( \lambda V_t \) and a log normal jump size. The stochastic variance \( V_t \) is given by

\[
dV_t = -\kappa_v (V_t - \bar{v}) dt + \sigma_v \sqrt{V_t}dB_{v,t}(Q),
\]

where the shocks,

\[
dB_{v,t}(Q) = \left( \rho dB_{1,t}(Q) + \sqrt{1-\rho^2}dB_{2,t}(Q) \right).
\]

Without loss of generality, we fix the time 0 and compute expectation at time \( T \). The cumulative log return in this period is,

\[
R_T = \int_0^T d\ln S_t dt = \int_0^T \left( r - \frac{1}{2}V_t \right) dt + \int_0^T \sqrt{V_t}dB_{1,t}(Q) + \int_0^T dJ_t - \int_0^T \mu^*\lambda V_t dt.
\]

The jump process in \( d\ln S_t \) is simply a Poisson compound process with jump intensity \( \lambda V_t \) and a normal distribution jump \( \sim N(\mu^Q, \sigma^2) \). The jump size \( \mu^* \) in \( dS_t \) satisfies,

\[
\mu^* = \exp(\mu^Q + \sigma^2/2) - 1
\]

The demeaned return is,

\[
R_T - E_0 [R_T] = - (\lambda \mu^* + 1/2) \int_0^T (V_t - E_0 [V_t]) dt + \int_0^T \sqrt{V_t}dB_{1,t} + \int_0^T dJ_t - \lambda \mu^Q \int_0^T E_0 [V_t] dt
\]

To compute the first and second moments, we define,

\[
X_T = \int_0^T \sqrt{V_t}dB_{1,t}, \quad Y_T = \int_0^T (V_t - E_0 [V_t]) dt,
\]

\[
J_T = \int_0^T dJ_t - \lambda \mu^Q \int_0^T E_0 [V_t] dt.
\]

The shocks in log return are hence separated into three terms
\[ R_T - \mathbb{E}_0[R_T] = X_T - (\lambda \mu^* + 1/2) Y_T + J_T \]

The last term \( J_T \), as the cumulative jump process, is independent from the shocks in Brownian motions, and hence independent from \( X_T, Y_T \). Its second and third moments is simply derived as,

\[
\mathbb{E}_0 [J_T^2] = \lambda \sigma_J^2 \int_0^T \mathbb{E}_0 [V_t] \, dt, \quad \mathbb{E}_0 [J_T^3] = \lambda \mu_J (\mu_J^2 + 3\sigma_J^2) \int_0^T \mathbb{E}_0 [V_t] \, dt
\]

Note that,

\[
\mathbb{E}_0 (V_t) = \bar{v} + (V_0 - \bar{v}) e^{-\kappa t}
\]

\[
V_t - \mathbb{E}_0 (V_t) = \sigma_v \int_0^t e^{-\kappa(t-u)} \sqrt{V_u} dB_u^v,
\]

hence,

\[
\int_0^T \mathbb{E}_0 [V_t] \, dt = \frac{1 - e^{-\kappa v T}}{\kappa v} (V_0 - \bar{v}) + \bar{v} T.
\]

The moments of \( J_T \) are solved.

\( X_T \) and \( Y_T \) has negative covariance and co-skewness due to the correlation between shocks in \( dV_t \) and \( dS_t \), we denote

\[ H_T = X_T - (\lambda \mu^* + 1/2) Y_T, \]

and compute the second and third moments.

The variance of \( H_T \) can be obtain by using Ito Isometry,

\[
\mathbb{E}_0 [H_T^2] = \int_0^T \mathbb{E}_0^Q [V_u] \, du
\]

\[- (2\lambda \mu^* + 1) \int_0^T \rho \sigma_v \frac{1 - e^{-\kappa_v (T-u)}}{\kappa_v} \mathbb{E}_0^Q [V_u] \, du
\]

\[ + (\lambda \mu^* + 1/2)^2 \int_0^T \sigma_v^2 \frac{(1 - e^{-\kappa_v (T-u)})^2}{\kappa_v^2} \mathbb{E}_0^Q [V_u] \, du \]

We now present the three terms as,

\[
\mathbb{E}_0 [H_T^2] = A - (2\lambda \mu^* + 1) B + (\lambda \mu^* + 1/2)^2 C,
\]

where,

\[ A = \int_0^T \mathbb{E}_0^Q [V_u] \, du, \]
\[
B = \int_0^T \rho \sigma_v \frac{1 - e^{-\kappa_v(T-u)}}{\kappa_v} E_Q^0 [V_u] \, du,
\]

\[
C = \int_0^T \sigma_v^2 \left( \frac{1 - e^{-\kappa_v(T-u)}}{\kappa_v^2} \right)^2 E_Q^0 [V_u] \, du.
\]

We now derive \(A, B, C\) respectively. For the interest of saving space, we leave out the middle steps and only present necessary equations. The first term, \(A = \int_0^T E_Q^0 [V_u] \, du\) is simple to compute,

\[
A = \int_0^T E_Q^0 [V_u] \, du = \frac{1 - e^{-\kappa_v T}}{\kappa_v} (V_0 - \bar{v}) + \bar{v} T.
\]

The second term, \(B = \int_0^T \rho \sigma_v \frac{1 - e^{-\kappa_v(T-u)}}{\kappa_v} E_Q^0 [V_u] \, du\), is presented as:

\[
B = \int_0^T \rho \sigma_v \frac{1 - e^{-\kappa_v(T-u)}}{\kappa_v} [e^{-\kappa_v u} (V_0 - \bar{v}) + \bar{v}] \, du.
\]

We split the this integral into two terms,

\[
B_1 = \int_0^T \rho \sigma_v \frac{1 - e^{-\kappa_v(T-u)}}{\kappa_v} e^{-\kappa_v u} (V_0 - \bar{v}) \, du,
\]

\[
B_2 = \int_0^T \rho \sigma_v \frac{1 - e^{-\kappa_v(T-u)}}{\kappa_v} \bar{v} u \, du.
\]

To simplify the calculation, we change the variable of the integral by letting \(x(u) = e^{\kappa_v u}\), such that \(du = \frac{1}{\kappa_v} x^{-1} \, dx\). Then the first term equals to,

\[
B_1 = \frac{\rho \sigma_v (V_0 - \bar{v})}{\kappa_v^2} \left( 1 - e^{-\kappa_v T} - e^{-\kappa_v T} (\kappa_v T) \right).
\]

Similarly, the second term in \(B\) equals,

\[
B_2 = \frac{\rho \sigma_v \bar{v}}{\kappa_v^2} (\kappa_v T - 1 + e^{-\kappa_v T}).
\]

\(B\) is the sum of \(B_1, B_2\).

Similarly, \(C\) is presented as,

\[
C = \int_0^T \sigma_v^2 \left( \frac{1 - e^{-\kappa_v(T-u)}}{\kappa_v^2} \right)^2 \left[ e^{-\kappa_v u} (V_0 - \bar{v}) + \bar{v} \right] \, du.
\]
We split it into two terms,

\[ C_1 = \int_0^T \sigma_v^2 \left( \frac{1 - e^{-\kappa_v(T-u)}}{\kappa_v^2} \right)^2 e^{-\kappa_v u} \left( V_0 - \bar{v} \right) du, \]

\[ C_2 = \int_0^T \sigma_v^2 \left( \frac{1 - e^{-\kappa_v(T-u)}}{\kappa_v^2} \right)^2 \bar{v} du. \]

We also use the change of variable method here to simplify the integral. \( C_1 \) equals to,

\[ C_1 = \frac{\sigma_v^2}{\kappa_v^3} \left( V_0 - \bar{v} \right) \left( 1 - 2e^{-\kappa_v T} - e^{-2\kappa_v T} \right). \]

\( C_2 \) equals to,

\[ C_2 = \frac{\sigma_v^2 \bar{v}}{\kappa_v^3} \left( \kappa_v T - 2 \left( 1 - e^{-\kappa_v T} \right) + 1/2 \left( 1 - e^{-2\kappa_v T} \right) \right). \]

The third term is the sum of \( C_1, C_2 \).

So far, we derived the second moments as the sum of \( E_0 \left[ H_T^2 \right] + E_0 \left[ J_T^2 \right] \). The third moments of \( H_T \) is more complicated, such that

\[ E_0 \left[ H_T^3 \right] = E_0 \left[ X_T^3 \right] - 3 \left( \lambda \mu^* + 1/2 \right) E_0 \left[ X_T^2 Y_T \right] + 3 \left( \lambda \mu^* + 1/2 \right)^2 E_0 \left[ X_T Y_T^2 \right] - \left( \lambda \mu^* + 1/2 \right)^3 E_0 \left[ Y_T^3 \right] \]

We now compute the above moments separately. Applying the Ito’s lemma on the \( X_T^3, X_T^2 Y_T, X_T Y_T^2, Y_T^3 \) respectively gives,

\[ E_0 \left( X_T^3 \right) = 3 \rho \sigma_v \int_0^T \frac{1 - e^{-\kappa_v (T-u)}}{\kappa_v} E_0 \left( V_u \right) du = 3B, \]

where \( B \) is the same as in the calculation for the second moment. The next moment is derived as,

\[ E_0 \left( X_T^2 Y_T \right) = \sigma_v^2 \int_0^T \left[ \left( \frac{1 - e^{-\kappa_v (T-u)}}{\kappa_v} \right)^2 + 2 \rho^2 \left( 1 - e^{-\kappa_v (T-u)} - \kappa_v (T-u)e^{-\kappa_v (T-u)} \right) \right] E_0 \left( V_u \right) du. \]

We compute this moment as,

\[ E_0 \left( X_T^2 Y_T \right) = \frac{1}{2 \kappa_v^2} \left( -e^{-2T \kappa_v} \left( 2 \left( \bar{v} - V_0 \right) + \bar{v} - 2e^{2T \kappa_v} (V_0 - \bar{v}) \left( 1 + 2 \rho^2 \right) + 4e^{T \kappa_v} \bar{v} (-1 + (-2 - T \kappa_v) \rho^2) \right) + e^{-T \kappa_v} \left( -2 \left( \bar{v} - V_0 \right) \left( 2 \rho^2 + T^2 \kappa_v \rho^2 + 2T \kappa_v (1 + \rho^2) \right) + e^{T \kappa_v} \bar{v} (-3 - 8 \rho^2 + 2T (\kappa_v + 2 \kappa_v \rho^2)) \right) \right) \sigma_v^2. \]
Similarly,

\[ E_0 (X_T Y_T^2) = \rho \sigma_v^3 \int_0^T \left[ \frac{2}{\kappa_v} \left( 1 - e^{-\kappa_v(T-u)} - \kappa_v(T-u)e^{-\kappa_v(T-u)} \right) \frac{1 - e^{-\kappa_v(T-u)}}{\kappa_v} \right] E_0 (V_u) \, du \]

\[ + \rho \sigma_v^3 \int_0^T \left[ \frac{1 - e^{-2\kappa_v(T-u)} - 2\kappa_v(T-u)e^{-\kappa_v(T-u)}}{\kappa_v^3} \right] E_0 (V_u) \, du \]

We compute this moment as,

\[ E_0 (X_T Y_T^2) = \frac{1}{\kappa_v} e^{-2T\kappa_v} \left( (-3V_0 + 2\bar{v} - T(2V_0 - \bar{v})\kappa_v) + 2e^{T\kappa_v}(-2 - T\kappa_v)(-2\bar{v} - T(-V_0 + \bar{v})\kappa_v) + e^{2T\kappa_v}(3V_0 + \bar{v}(-10 + 3T\kappa_v)) \right) \rho \sigma_v^3. \]

In the end,

\[ E_0 (Y_T^3) = 3\sigma_v^4 \int_0^T \left[ \frac{1 - e^{-2\kappa_v(T-u)} - 2\kappa_v(T-u)e^{-\kappa_v(T-u)}}{1 - e^{-\kappa_v(T-u)}} \right] E_0 (V_u) \, du. \]

We compute as,

\[ E_0 (Y_T^3) = \frac{1}{2\kappa_v^5} \left( e^{-T\kappa_v} \left( 2e^{T\kappa_v}\bar{v}(-8 + 3T\kappa_v) - 3(V_0 - \bar{v})(-1 + 2T\kappa_v + 2T^2\kappa_v^2) \right) + e^{-3T\kappa_v} \right) \]

\[ \left( (-3V_0 + \bar{v}) + 6e^{3T\kappa_v}(V_0 - \bar{v}) + 6e^{T\kappa_v}(\bar{v} + T\kappa_v + V_0(-1 - 2T\kappa_v)) + 6e^{2T\kappa_v}(\bar{v}(3 + 2T\kappa_v)) \right) \sigma_v^4. \]