Option-Implied Crash Index

Jun Pan

Shanghai Advanced Institute of Finance (SAIF)
Shanghai Jiao Tong University

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Introduction

- The seminal work of Black, Scholes, and Merton:
  - Pure diffusion: Black and Scholes (1973), Merton (1973).
  - Jump diffusion: Merton (1976).
  - Pure jump: Cox and Ross (1976).
- Models incorporating stochastic volatility and jump risk:
  - Stochastic volatility:
    Hull and White (1987), Stein and Stein (1991), Heston (1993).
  - Jump-diffusion models:
    Bakshi, Cao, and Chen (1998), Bates (2000), Duffie, Pan and Singleton (2000).
- Empirical evidence:
  - Using cross-sectional options: Bakshi, Cao, and Chen (1998).
  - Using joint-times series of options and stock prices: Pan (2002).

Models, Data, and Risk Measures

- The complete-market setting of Black, Merton, and Scholes:
  - Model parameter: the diffusion coefficient σ.
  - Application: the option-implied volatility index (VIX).
- The jump-diffusion models with the jump component of Merton (1976):
  - Model parameter: the mean jump size μ.
  - Application: the option-implied crash index (CIX).
- The non-parametric approach of Breeden and Litzenberger (1978):
  - Ait-Sahalia and Lo (1998): the state-price density.
  - CBOE and Goldman Sachs (2003): the updated VIX index.
  - Bakshi, Kapadia, and Madan (2003): option-implied skew.

The Stochastic Volatility Model with Jump (SVJ)

- The data generating process:

$$dS_t = \left(r_t - q_t + \boxed{\eta^s V_t} + \boxed{\lambda V_t \left(\mu - \mu^*\right)} \right) S_t \, dt + \sqrt{V_t} S_t \, dW_t^{(1)} + \boxed{dZ_t - \mu S_t \lambda V_t \, dt}$$

$$dV_t = \kappa_v \left(\bar{v} - V_t\right) dt + \sigma_v \sqrt{V_t} \left(\rho \, dW_t^{(1)} + \sqrt{1-\rho^2} \, dW_t^{(2)} \right)$$

- The risk-neutral dynamics:

$$dS_t = \left(r_t - q_t\right) S_t \, dt + \sqrt{V_t} S_t \, dW_t^{(Q1)} + \boxed{dZ_t^Q - \mu^* S_t \lambda V_t \, dt}$$

$$dV_t = \left(\kappa_v \left(\bar{v} - V_t\right) + \boxed{\eta^v V_t} \right) dt + \sigma_v \sqrt{V_t} \left(\rho \, dW_t^{(Q1)} + \sqrt{1-\rho^2} \, dW_t^{(Q2)} \right)$$

- The market prices of risks:
  - The diffusion risk premium in index returns: $\eta^s \, V_t$.
  - The jump risk premium in index returns: $\lambda \, V_t \left(\mu - \mu^*\right)$.
  - The volatility risk premium: $\eta^v \, V_t$.

Option Pricing Under the SVJ Model

- The time-t price of a European-style call option:

$$C_t^{\mathsf{SVJ}} = \mathrm{E}_t^Q \left[\exp\left(-\int_t^T r_u \, \mathrm{d}u \right) (S_T - K)^+ \right] = S_t \, f\left(V_t, \vartheta, r_t, q_t, \tau, \frac{K}{S_t} \right)$$

  - K is the strike price and $T = t + \tau$ is the expiration date.
  - The latent state variable: V_t.
  - The model parameters: $\vartheta = (\kappa_v, \bar{v}, \sigma_v, \rho, \lambda, \mu, \sigma_J, \eta^s, \eta^v, \mu^*)$.
- Via put/call parity, the time-t price of a European-style put option:

$$P_t^{\mathsf{SVJ}} = S_t \, f\left(V_t, \vartheta, r_t, q_t, \tau, \frac{K}{S_t} \right) - S_t \, e^{-q\tau} + K \, e^{-r\tau} \, .$$

Model Estimation

- We use the joint time-series of the S&P 500 index and options (Pan 2002).
- The moment conditions are constructed on the two state variables:
  - Daily index returns: $y_t = \ln S_t - \ln S_{t-1} - r_t - q_t$
  - Time-t volatility V_t, inferred from an ATM call, given model parameters ϑ,

$$C_t^{\text{Market}} = C_t^{\text{SVJ}} = S_t f\left(V_t, \vartheta, r_t, q_t, \tau, \frac{K}{S_t}\right) .$$

- The model parameters ϑ and the latent V_t are simultaneously estimated,
  - using moment conditions based on

$$\begin{aligned}
\varepsilon_t^{y1} &= y_t - M_1\left(V_{t-1}, \vartheta\right) & \qquad \varepsilon_t^{v1} &= V_t - M_5\left(V_{t-1}, \vartheta\right) \\
\varepsilon_t^{y2} &= y_t^2 - M_2\left(V_{t-1}, \vartheta\right) & \qquad \varepsilon_t^{v2} &= V_t^2 - M_6\left(V_{t-1}, \vartheta\right) \\
\varepsilon_t^{y3} &= y_t^3 - M_3\left(V_{t-1}, \vartheta\right) & \qquad \varepsilon_t^{yv} &= y_t V_t - M_7\left(V_{t-1}, \vartheta\right) \\
\varepsilon_t^{y4} &= y_t^4 - M_4\left(V_{t-1}, \vartheta\right) ,
\end{aligned}$$

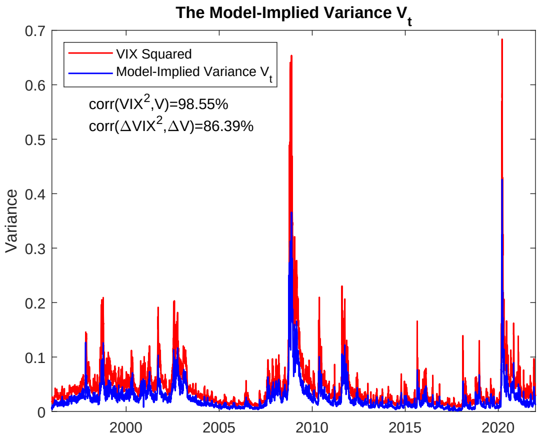
  - and pricing errors of 30-day ITM calls and 90-day ATM calls.

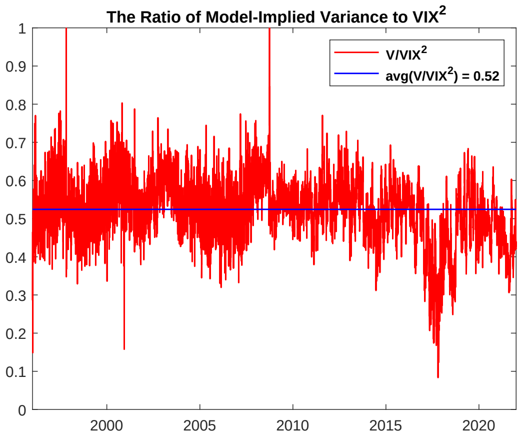
Estimation Results: Model Parameters

Panel A: SVJ estimation using 1996-2021 daily price data									
k_v	$\bar{v}$	σ_v	ρ	λ	μ	σ_J	η_s	η_v	μ^*
5.88	0.0108	0.29	-0.52	23.9	-1.00%	2.97%	3.10	3.19	-16.16%
[45.12]	[8.70]	[15.54]	[-15.11]	[13.71]	[-0.14]	[5.83]	[1.51]	[16.65]	[-37.32]

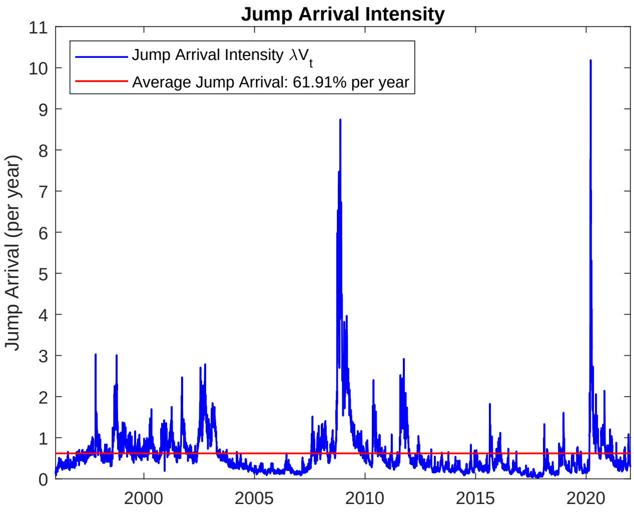
Panel B: SVJ fitting the joint moments of y_t and V_t						
y_t	y_t^2	y_t^3	y_t^4	V_t	V_t^2	$y_t V_t$
20.14	0.35	1.15	-0.20	0.03	0.01	1.10
[0.26]	[0.09]	[0.21]	[-0.06]	[0.04]	[0.02]	[0.20]

Estimation Results: The Latent State Variable V_t

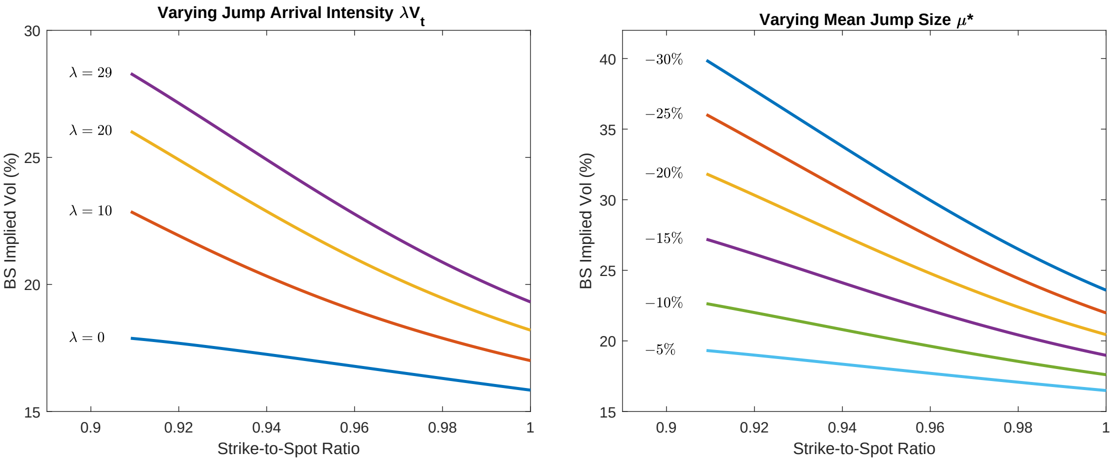




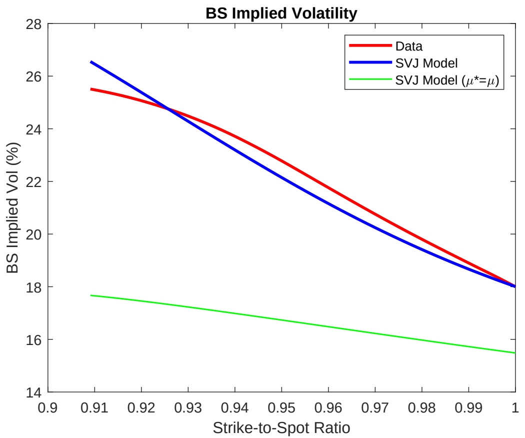
Estimation Results: Model-Implied Jump Arrival Intensity

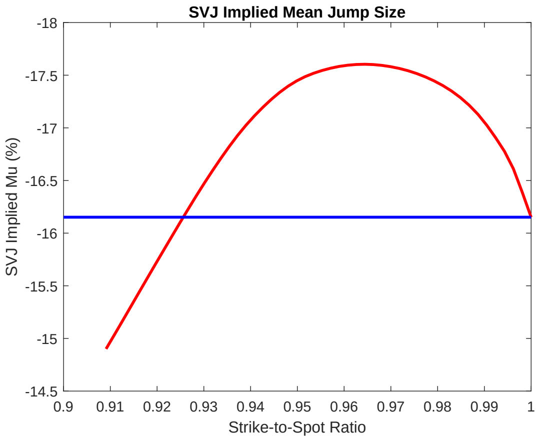


Estimation Results: Model-Implied Volatility Surface



Option-Implied Crash Surface





Vol Surface, Data vs Model

Option-Implied Crash Surface

Construct the Option-Implied Crash Index (CIX)

- Infer the SVJ model implied $\mu_t^I(\tau, K)$ from an OTM put with market price P_t^{Market},

$$P_t^{\text{Market}} = P_t^{\text{SVJ}} = S_t\, f\left(V_t, \vartheta^{\perp}, \mu_t^I(\tau, K), r, q, \tau, \frac{K}{S_t}\right) - S_t\, e^{-q\tau} + K\, e^{-r\tau}\,,$$

where we fix V_t and all other parameters $\vartheta^{\perp}$ to the estimation results.

- This parallels the Black-Scholes implied $\sigma_t^I(\tau, K)$,

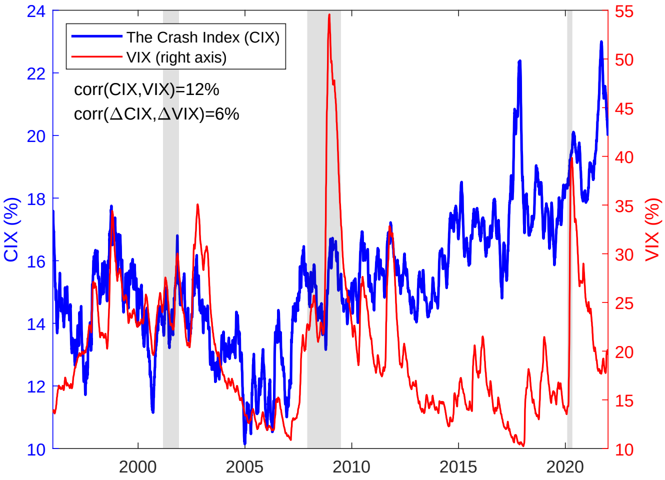
$$P_t^{\text{Market}} = P_t^{\text{BS}} = S_t\, f\left(\sigma_t^I(\tau, K), r, q, \tau, \frac{K}{S_t}\right) - S_t\, e^{-q\tau} + K\, e^{-r\tau}\,.$$

- Following VIX, we construct CIX by interpolating around the 30-day to expiration,

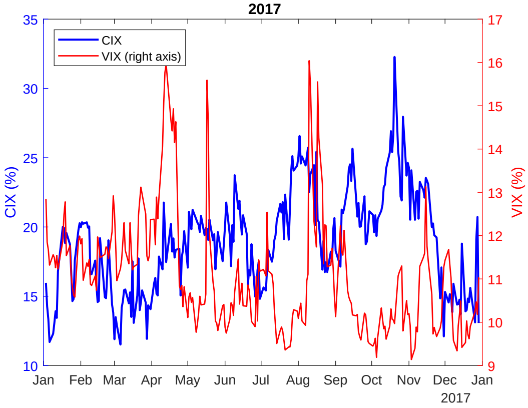
$$-\text{CIX}_t = \mu_t^I(\tau_1)\,\frac{\tau_2 - 30}{\tau_2 - \tau_1} + \mu_t^I(\tau_2)\,\frac{30 - \tau_1}{\tau_2 - \tau_1}\,,$$

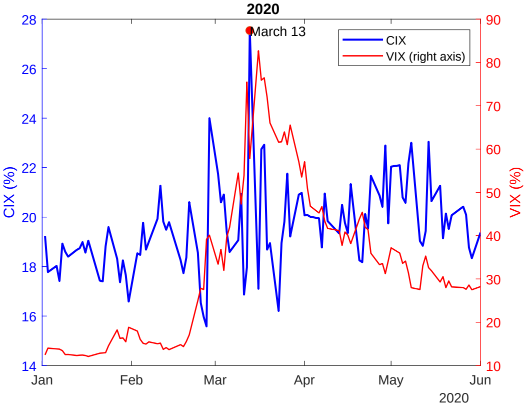
using $\tau_1 \leq 30 < \tau_2$ and $\mu_t^I(\tau)$ for each τ is averaged across all $K/S \in [0.93, 0.97]$.

Time-Series of CIX vs VIX



Time-Series of CIX vs VIX





Validating CIX using Non-Parametric Skew

- Since 2003, the CBOE VIX is computed via

$$e^{-r\tau}\mathrm{E}_t^Q\left(R(t,\tau)^2\right) = \int_{S_t}^{\infty} \frac{2\left(1-\ln\left(\frac{K}{S_t}\right)\right)}{K^2} C_t(\tau,K)dK + \int_0^{S_t} \frac{2\left(1+\ln\left(\frac{S_t}{K}\right)\right)}{K^2} P_t(\tau,K)dK\,.$$

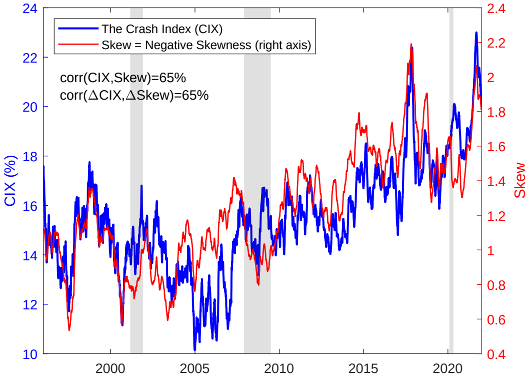
where $R(t,\tau) = \ln S_{t+\tau} - \ln S_t$.

- Bakshi, Kapadia, and Madan (2003) computes the risk-neutral skewness via

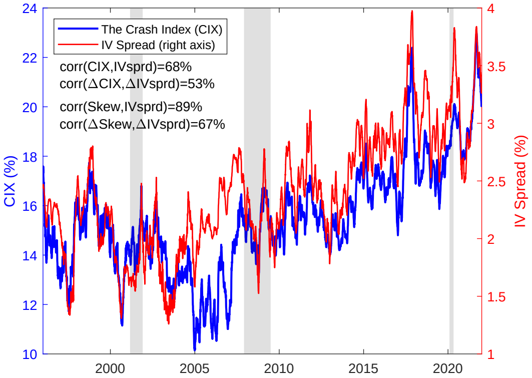
$$e^{-r\tau}\mathrm{E}_t^Q\left(R(t,\tau)^3\right) = \int_{S_t}^{\infty} \frac{6\ln\left(\frac{K}{S_t}\right) - 3\left(\ln\left(\frac{K}{S_t}\right)\right)^2}{K^2} C_t(\tau,K)dK - \int_0^{S_t} \frac{6\ln\left(\frac{S_t}{K}\right) + 3\left(\ln\left(\frac{S_t}{K}\right)\right)^2}{K^2} P_t(\tau,K)dK\,.$$

  - Crashes: important in generating negative skewness.
  - The non-parametric skewness can be used to validate the information content of our CIX index.

Time-Series of CIX, Skew and IV Spread

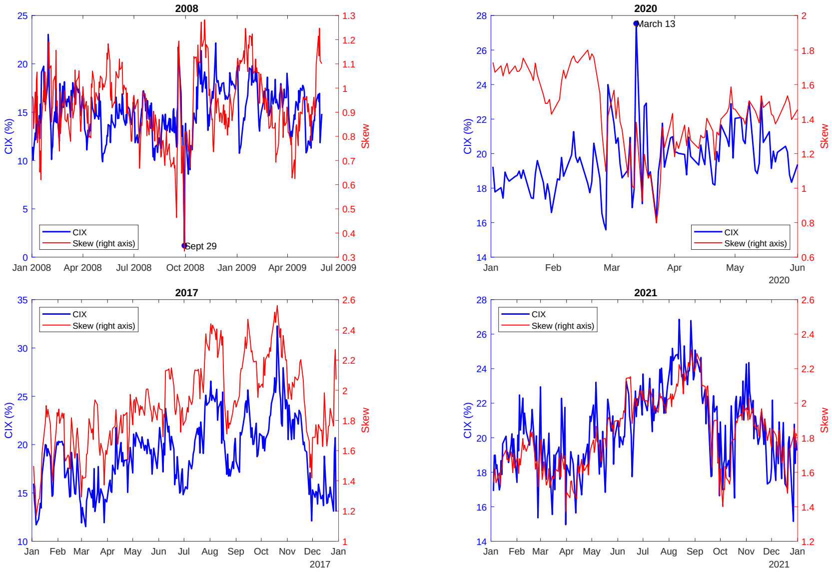


CIX and Skewness

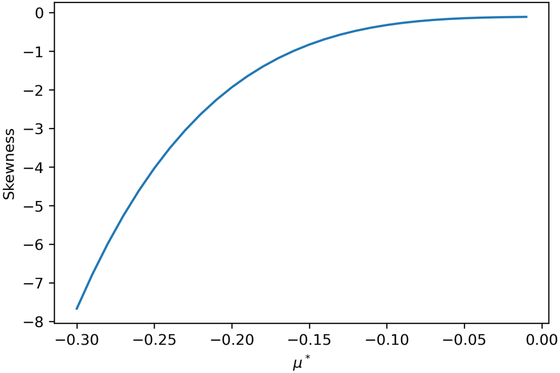


CIX and IV Spread

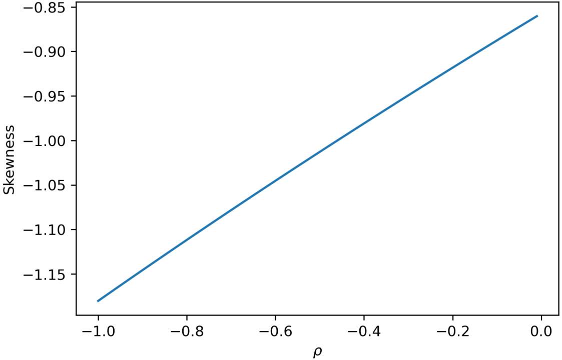
Time-Series of CIX and Skewness



Skewness under the SVJ Model

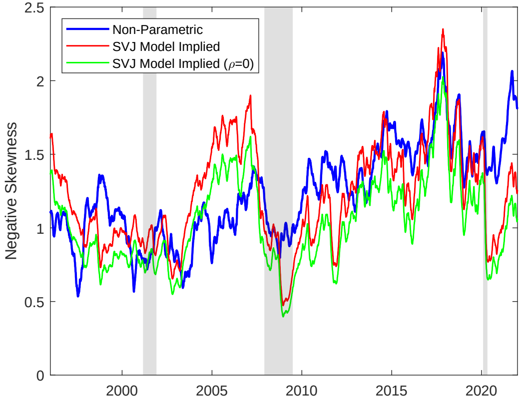


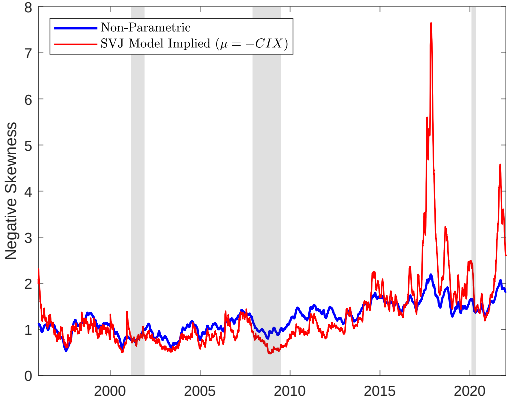
SVJ Skewness with Varying μ^*



SVJ Skewness with Varying ρ

Time-Series of Skewness, Non-Parametric vs the SVJ Model





Explaining the Crash Index (CIX)

	Dependent Variable: ΔCIX_t (%)							
ΔVIX_t	6.37 [3.30]				11.38 [7.63]		10.61 [7.07]	10.50 [7.19]
$\Delta IVsprd_t$		59.42 [29.39]			32.32 [14.49]		32.16 [14.14]	32.12 [14.10]
$\Delta Skew_t$			64.60 [31.82]		46.37 [15.37]		46.14 [15.18]	46.31 [15.25]
$\Delta(P/C)_t$				5.53 [3.66]	3.25 [2.92]		3.15 [2.85]	2.99 [2.71]
$\Delta Noise_t$						2.58 [1.66]		2.51 [2.30]
$\Delta Term_t$						-2.70 [-1.95]		-4.00 [-3.32]
ΔTED_t						-1.53 [-1.09]		-0.38 [-0.33]
$\Delta Dsprd_t$						1.03 [0.68]		-0.31 [-0.24]
CIX_{t-1}							-0.35 [-0.34]	-0.39 [-0.38]
VIX_{t-1}							0.01 [0.01]	0.29 [0.27]
Ret_{t-1}							-2.65 [-2.67]	-2.49 [-2.52]
$1_{3rd\ Friday}$							1.72 [2.38]	1.41 [1.95]
adj R^2 (%)	0.39	35.29	41.72	0.29	50.03	0.10	50.10	50.26

Return Predictability Using Option-Implied Risk Measures

Average Daily SPX Returns (bps)									
	following large increases					following large reductions			
Prob(Tail)	CIX	VIX	Skew	IVSprd	Prob(Tail)	CIX	VIX	Skew	IVSprd
2%	-47.41	59.88	-29.08	-23.87	2%	20.28	-4.99	25.22	35.70
	[-2.89]	[2.03]	[-2.03]	[-1.47]		[1.70]	[-0.26]	[1.46]	[1.92]
5%	-28.96	29.73	-24.84	-12.63	5%	19.55	-3.39	21.32	30.17
	[-3.39]	[2.14]	[-3.27]	[-1.42]		[2.47]	[-0.34]	[2.33]	[2.99]
10%	-18.29	17.52	-13.31	-9.76	10%	11.60	4.25	22.08	21.59
	[-3.44]	[2.12]	[-2.56]	[-1.66]		[2.24]	[0.71]	[3.93]	[3.52]
15%	-12.16	14.04	-12.23	-6.53	15%	8.32	3.72	19.13	18.99
	[-2.89]	[2.36]	[-3.10]	[-1.46]		[2.00]	[0.83]	[4.49]	[4.01]
20%	-10.41	12.06	-10.42	-4.77	20%	6.64	2.97	15.77	17.29
	[-2.90]	[2.52]	[-3.09]	[-1.26]		[1.92]	[0.82]	[4.40]	[4.45]

Return Predictability Using Option-Implied Risk Measures

	Dependent Variable: Day $t+1$ SPX Returns (bps)						
1_{ΔCIX_t}	-24.65			-20.20	-23.34		-3.59
	[-4.39]			[-4.27]	[-2.90]		[-2.34]
1_{ΔVIX_t}		15.14		22.62			
		[2.32]		[3.08]			
1_{ΔSKEW_t}			-19.11		-13.11		
			[-3.71]		[-2.01]		
$1_{\Delta \text{CIX}_t} \times 1_{\Delta \text{VIX}_t}$				-36.98			
				[-1.51]			
$1_{\Delta \text{CIX}_t} \times 1_{\Delta \text{SKEW}_t}$					9.16		
					[0.68]		
ΔCIX_t						-6.29	-4.11
						[-4.20]	[-2.44]
constant	6.36	2.37	5.81	4.25	7.07	3.21	4.40
	[4.76]	[1.75]	[4.46]	[3.07]	[5.13]	[3.08]	[3.89]
$R^2(\%)$	0.37	0.14	0.22	0.65	0.43	0.40	0.48

Conclusions

- We construct an option-implied Crash Index (CIX) by exploring the pricing difference between the OTM puts and ATM options, benchmarked against the SVJ model.
- The construction of our CIX index is analogous to that of the VIX index:
  - The mean jump size μ of the SVJ model implied by OTM puts.
  - The volatility parameter σ of the Black-Scholes model implied by ATM options.
- Empirically, we find that
  - The CIX index is closely related to the non-parametric option-implied skewness and positively correlated with the put/call volume ratio.
  - Post 2008, the CIX index has increased significantly.
  - Large increases in CIX are followed by large negative SPX returns.
  - By contrast, large increases in VIX are followed by large positive SPX returns.