An Equilibrium Model of Rare Event Premia

Jun Liu
The Anderson School at UCLA

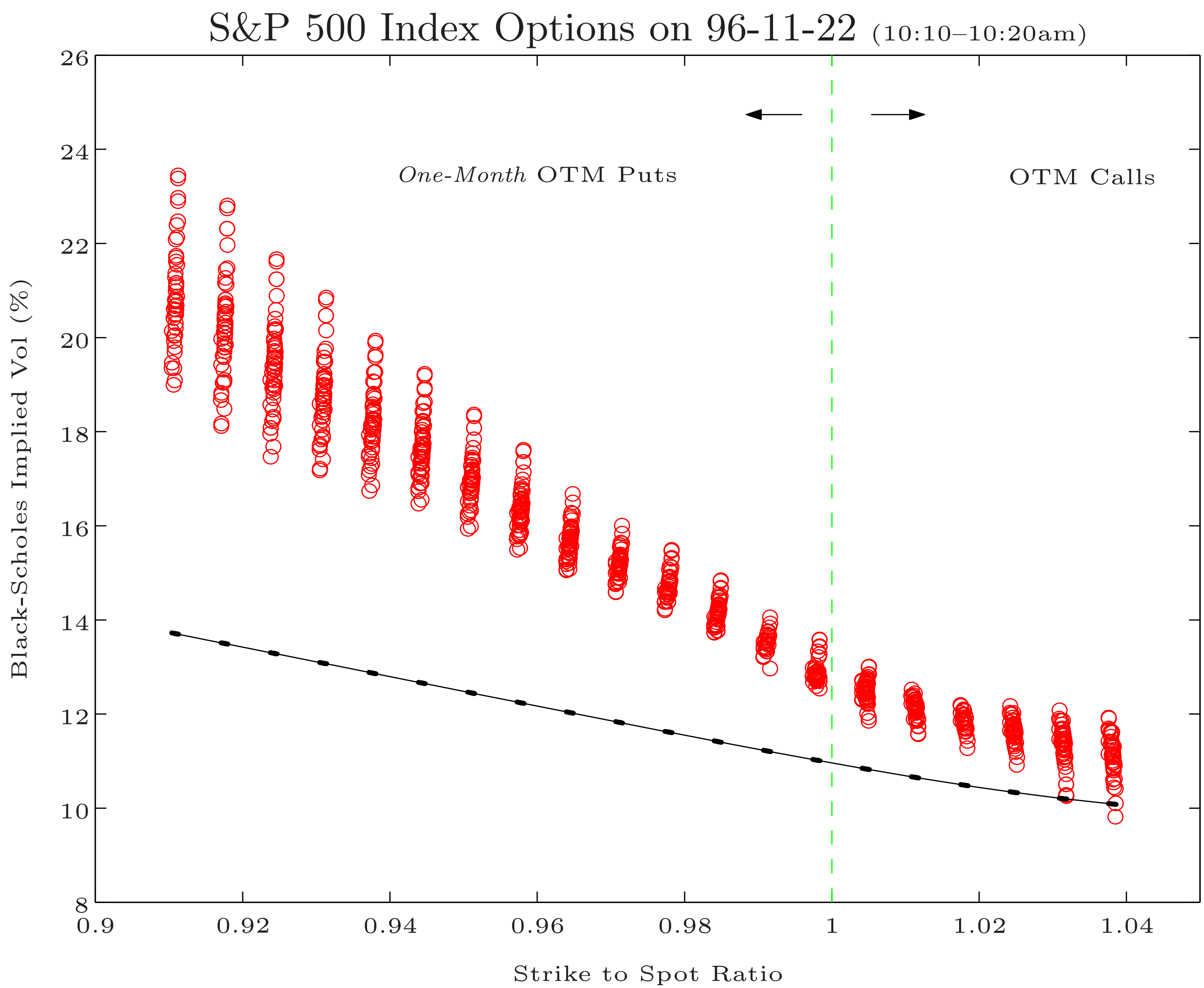
Jun Pan
MIT Sloan School of Management

Tan Wang
University of British Columbia

Introduction

This paper studies the asset pricing implication of imprecise information about rare events.

- Rare events: small probability, sudden occurrence, high impact.
- Models with rare events are hard to estimate with precision.
- How do investors deal with this aspect of model uncertainty?
- Testable implications — some securities are more sensitive to rare events than others.



S&P 500 Index Options on 96-11-22 (10:10–10:20am)
One-Month OTM Puts
OTM Calls
Black-Scholes Implied Vol (%)
Strike to Spot Ratio
26
24
22
20
18
16
14
12
10
8
0.9
0.92
0.94
0.96
0.98
1
1.02
1.04

Related Literature

- The conceptual differentiation between risk and uncertainty: Knight (1921) and Ellsberg (1961).
- Multi-prior expected utility: Gilboa and Schmeidler (1989), Epstein and Wang (1994), Chen and Epstein (2000).
- Robust control: Anderson, Hansen and Sargent (2000), Maenhout (1999), Uppal and Wang (2000).
- Other possible explanations for the differential pricing of diffusive and jump risks:
  - crash aversion [Bates (2001)]
  - disappointment aversion [Gul (1991)]
  - loss aversion [Kahneman and Tversky (1979)]
  - Bayesian learning about the jump component (?)

An Economy with Rare Events

A pure-exchange economy with one representative agent and one perishable consumption good. The aggregate endowment of the consumption good [Naik and Lee (1990)]:

$$dY_t = \mu\, Y_t\, dt + \sigma Y_t\, dB_t + \left(e^{Z_t} - 1\right) Y_{t-}\, dN_t\,.$$

- The diffusive risk is controlled by the Brownian shock dB
- The jump timing is dictated by the Poison process N_t
- Given arrival at time t, the jump size is controlled by Z_t

The Reference Model

- The model parameters:
  - The mean growth rate μ and volatility σ without jumps.
  - The jump-arrival intensity λ.
  - The mean percentage jump size $k = E\left(\exp(Z) - 1\right)$.
  - The variance of jump size $\sigma_J^2 = \text{var}(Z)$.
- For the purpose of making investment and consumption choices, the investor needs to know the values of the model parameters, which he can estimate using the exisiting data.
- Let P be this reference model that best fits the existing data.

Imprecise Knowledge about Rare Events

- The jump component of the reference model is not reliable:
  - typically noisy due to the infrequent nature of jumps.
  - could be biased because of the "Peso" problem.
- Knowing that the reference model P is not entirely reliable, the investor decides to "test drive" alternative models.
- Not any models — only those that are different in the jump component. In particular, we focus on the imprecise knowledge about the jump-timing λ and jump-size k.

The Set of Alternative Models

- Each alternative model $P(\xi)$ is defined relative to the reference model P by its Radon-Nikodym derivative $\xi_T = dP(\xi)/dP$:

$$d\xi_t = \left(e^{a+b\,Z_t-b\,\mu_J-\frac{1}{2}b^2\sigma_J^2} - 1\right)\xi_{t-}\,dN_t - (e^a - 1)\,\lambda\,\xi_t\,dt\,.$$

- For example, if $a = b = 0$, then $\xi = 1$ and the alternative model is the same as the reference model.

- A more general pair $a, b \in \mathbb{R}$ produces an alternative model that is different from the reference model. But the difference is only with respect to the jump likelihood and magnitude:

$$\lambda^\xi = \lambda\,e^a \quad \text{and} \quad k^\xi = (1+k)\,e^{b\,\sigma_J^2} - 1\,.$$

- This collection of alternative models is denoted by $\mathcal{P}$.

Robust Control for Rare Events

- Let U_t be the investor's utility at time t. In a standard setting,

$$U_t = \frac{c_t^{1-\gamma}}{1-\gamma} + e^{-\rho} E_t \left(U_{t+1} \right),$$

  taking expectation with respect to the reference model P.

- Knowing that P is not reliable, the investor thinks about tomorrow differently:

$$e^{-\rho} \inf_{P(\xi) \in \mathcal{P}} \left\{ E_t^{\xi} \left(U_{t+1} \right) + \frac{1}{\phi} \psi \left(U_t \right) \, E_t^{\xi} \left[h \left(\ln \frac{\xi_{t+1}}{\xi_t} \right) \right] \right\}$$

  - evaluates his future prospect under all possible $P(\xi) \in \mathcal{P}$.
  - penalizes his choice of $P(\xi)$ by its deviation from P.
  - the penalty function: $h(x) = x + \beta \left(e^x - 1 \right)$.

- The constant $\phi > 0$ captures his degree of uncertainty aversion.

The Equilibrium Equity Premium

The total equity premium has three components:

1. the diffusive risk premium $= \gamma\,\sigma^2$
2. the jump risk premium $= \lambda\,k - \bar{\lambda}\,\bar{k}$

$$\bar{\lambda} = \lambda\,(1+k)^{-\gamma}\; e^{\frac{1}{2}\gamma(1+\gamma)\,\sigma_J^2}\,, \quad \bar{k} = (1+k)\,e^{-\gamma\,\sigma_J^2} - 1\,.$$

3. the rare event premium $= \bar{\lambda}\,\bar{k} - \lambda^Q\,k^Q$

$$\lambda^Q = \bar{\lambda}\,e^{a^* - \gamma\, b^* \sigma_J^2}\,, \quad k^Q = \left(1+\bar{k}\right) e^{b^* \sigma_J^2} - 1\,,$$

where a^* and b^* are the optimal parameters determined by ϕ.

Observationally Equivalent?

As a rough calibration, let's set the diffusive volatility $\sigma = 15\%$, jump arrival intensity $\lambda = 1/3$, and random jump sizes with mean $\mu_J = -1\%$ and standard deviation $\sigma_J = 4\%$.

The observation that the total equity premium is 8% per year can be supported by various pairs of γ and ϕ:

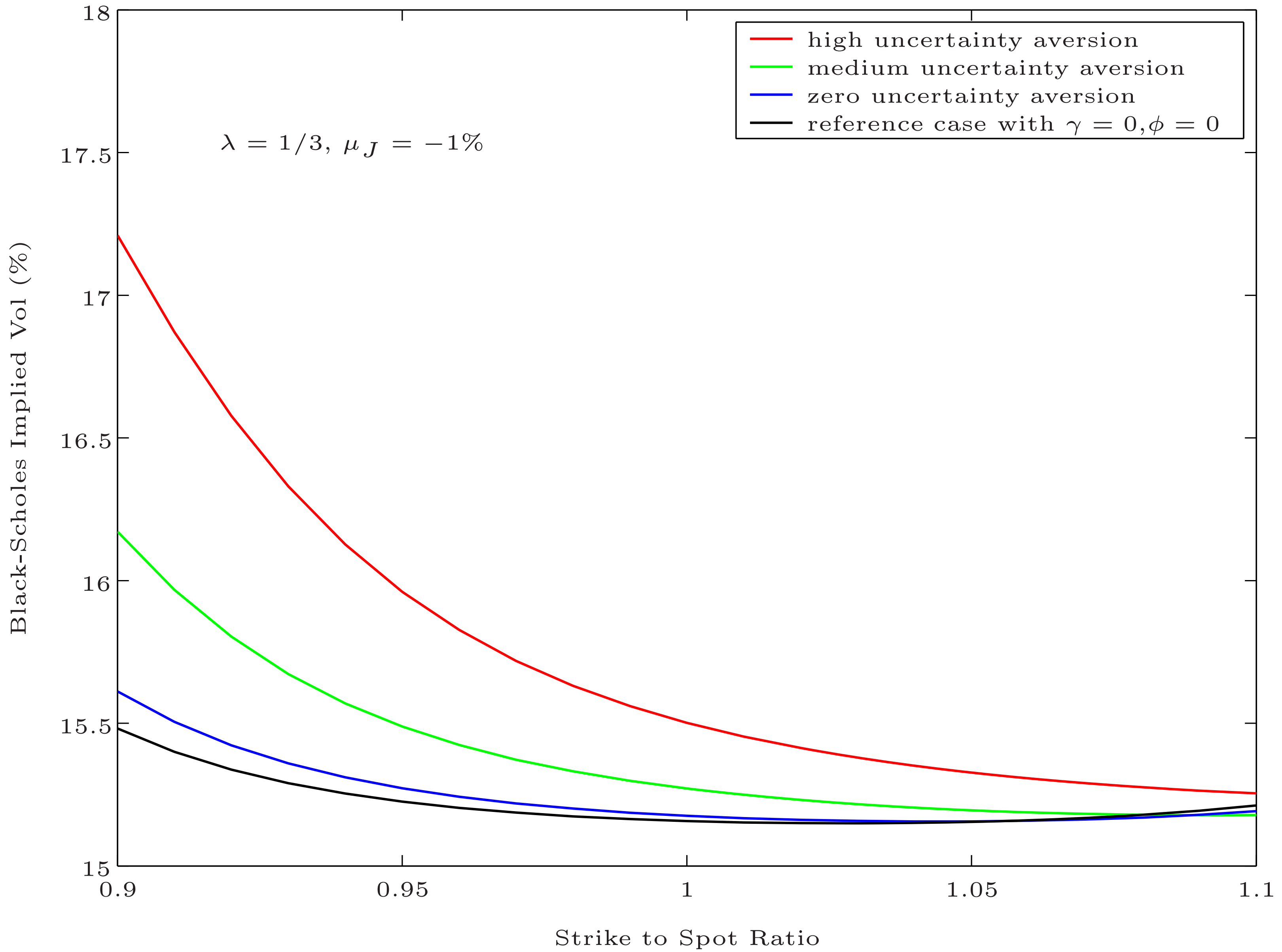
- an investor with $\gamma = 3.47$ and $\phi = 0$.
- an investor with $\gamma = 3.15$ and $\phi = 10$.
- an investor with $\gamma = 2.62$ and $\phi = 20$.
- $\ldots$.

For all these cases, our model implies very different compositions of the total equity premium:

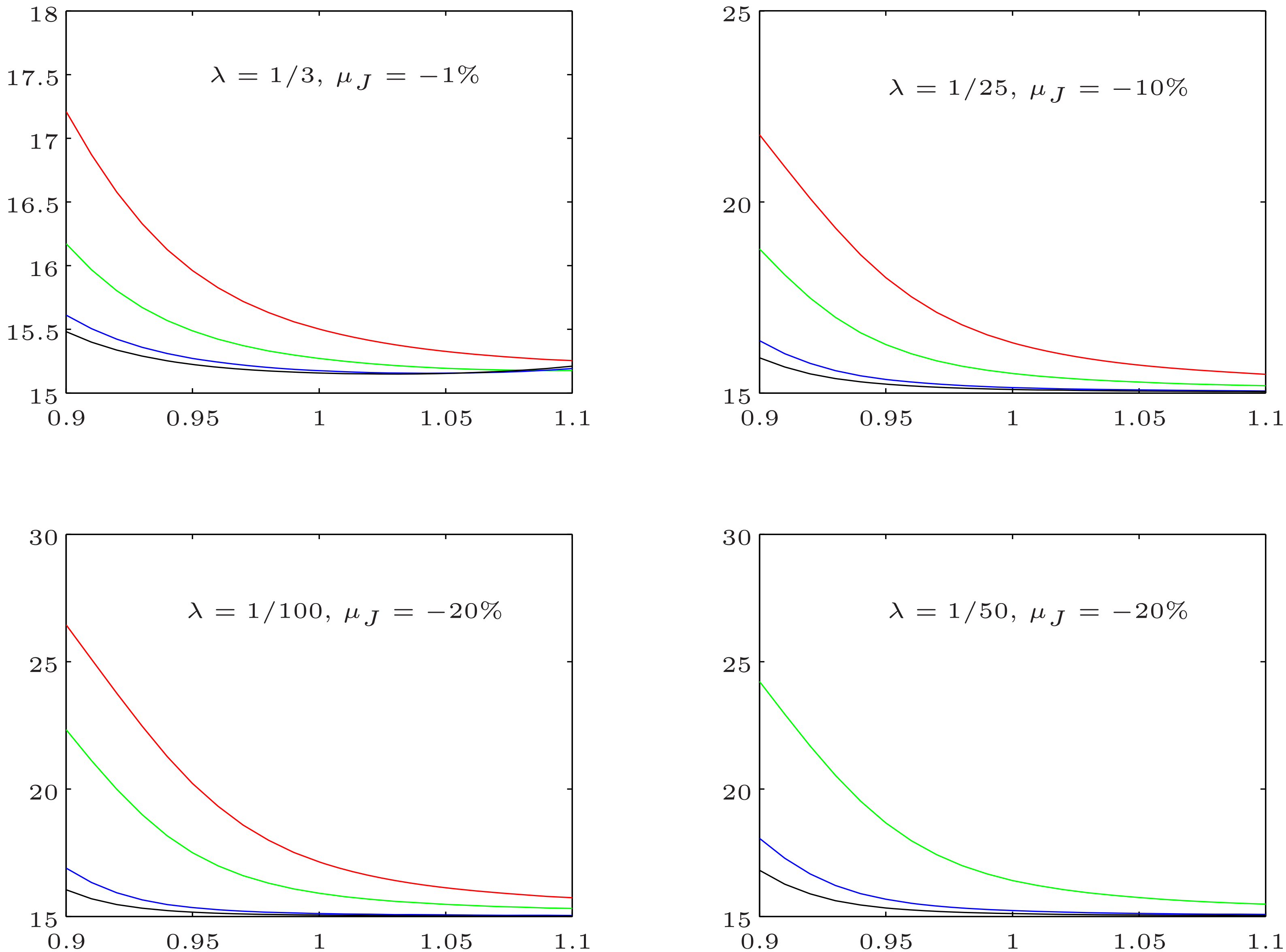
		risk premia		uncertainty	total equity
ϕ	γ	diffusive	jump	premium	premium
0	3.47	7.80%	0.20%	0	8%
10	3.15	7.09%	0.19%	0.72%	8%
20	2.62	5.91%	0.15%	1.94%	8%

Testable Implications on Options

- Unlike equity, options are non-linear in nature, with different sensitivities to diffusive shocks and jumps.
- In fact, by offering options with different degrees of moneyness and maturity, the options market provides a rich spectrum of such differential sensitivities.
- For this reason, the options market provides an ideal place for us to test our model's prediction on rare event premia.



high uncertainty aversion
medium uncertainty aversion
zero uncertainty aversion
reference case with $\gamma = 0, \phi = 0$
$\lambda = 1/3,\ \mu_J = -1\%$
Black-Scholes Implied Vol (%)
Strike to Spot Ratio
18
17.5
17
16.5
16
15.5
15
0.9
0.95
1
1.05
1.1



$\lambda = 1/3,\ \mu_J = -1\%$
$\lambda = 1/25,\ \mu_J = -10\%$
$\lambda = 1/100,\ \mu_J = -20\%$
$\lambda = 1/50,\ \mu_J = -20\%$

Conclusion

- We modified the standard pure-exchange economy by adding jumps as rare events, and by allowing the representative agent to perform robust control as a precaution against possible model mis-specification with respect to rare events.
- Provided an explicitly solved equilibrium, and showed that the total equity premium has three components: the diffusive risk premium, the jump risk premium, and the rare event premium.
- Examined the testable implications of our model on the options market, and documented the importance of the uncertainty aversion toward rare events in explaining the options data.