An Equilibrium Model of Rare Event Premia

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Introduction

This paper studies the asset pricing implication of imprecise information about rare events.

- Rare events: small probability, sudden occurrence, high impact.
- Models with rare events are hard to estimate with precision.
- How do investors deal with this aspect of model uncertainty?
- Testable implications — some securities are more sensitive to rare events than others.
S&P 500 Index Options on 96-11-22 (10:10 – 10:20am)

One-Month OTM Puts

OTM Calls

Strike to Spot Ratio

Black-Scholes Implied Vol (%)
Related Literature

- The conceptual differentiation between risk and uncertainty: Knight (1921) and Ellsberg (1961).
- Other possible explanations for the differential pricing of diffusive and jump risks:
  - crash aversion [Bates (2001)]
  - disappointment aversion [Gul (1991)]
  - loss aversion [Kahneman and Tversky (1979)]
  - Bayesian learning about the jump component (?)
An Economy with Rare Events

A pure-exchange economy with one representative agent and one perishable consumption good. The aggregate endowment of the consumption good [Naik and Lee (1990)]:

\[ dY_t = \mu Y_t \, dt + \sigma Y_t \, dB_t + \left( e^{Z_t} - 1 \right) Y_{t-} \, dN_t. \]

- The diffusive risk is controlled by the Brownian shock \( dB \)
- The jump timing is dictated by the Poison process \( N_t \)
- Given arrival at time \( t \), the jump size is controlled by \( Z_t \)
The Reference Model

- The model parameters:
  - The mean growth rate $\mu$ and volatility $\sigma$ without jumps.
  - The jump-arrival intensity $\lambda$.
  - The mean percentage jump size $k = E(\exp(Z) - 1)$.
  - The variance of jump size $\sigma_j^2 = \text{var}(Z)$.

- For the purpose of making investment and consumption choices, the investor needs to know the values of the model parameters, which he can estimate using the existing data.

- Let $P$ be this reference model that best fits the existing data.
Imprecise Knowledge about Rare Events

• The jump component of the reference model is not reliable:
  – typically noisy due to the infrequent nature of jumps.
  – could be biased because of the “Peso” problem.

• Knowing that the reference model $P$ is not entirely reliable, the investor decides to “test drive” alternative models.

• Not any models — only those that are different in the jump component. In particular, we focus on the imprecise knowledge about the jump-timing $\lambda$ and jump-size $k$. 
The Set of Alternative Models

- Each alternative model $P(\xi)$ is defined relative to the reference model $P$ by its Radon-Nikodym derivative $\xi_T = dP(\xi)/dP$:

  \[ d\xi_t = \left( e^{a+bZ_t-b\mu_j-\frac{1}{2}b^2\sigma_j^2} - 1 \right) \xi_t - dN_t - (e^a - 1) \lambda \xi_t dt. \]

- For example, if $a = b = 0$, then $\xi = 1$ and the alternative model is the same as the reference model.

- A more general pair $a, b \in \mathbb{R}$ produces an alternative model that is different from the reference model. But the difference is only with respect to the jump likelihood and magnitude:

  \[ \lambda^\xi = \lambda e^a \quad \text{and} \quad k^\xi = (1 + k) e^{b\sigma_j^2} - 1. \]

- This collection of alternative models is denoted by $\mathcal{P}$. 
Robust Control for Rare Events

- Let $U_t$ be the investor’s utility at time $t$. In a standard setting,

$$U_t = \frac{c_t^{1-\gamma}}{1-\gamma} + e^{-\rho} E_t (U_{t+1}),$$

taking expectation with respect to the reference model $P$.

- Knowing that $P$ is not reliable, the investor thinks about tomorrow differently:

$$e^{-\rho} \inf_{P(\xi) \in \mathcal{P}} \left\{ E_t^\xi (U_{t+1}) + \frac{1}{\phi} \psi (U_t) E_t^\xi \left[ h \left( \ln \frac{\xi_{t+1}}{\xi_t} \right) \right] \right\}$$

- evaluates his future prospect under all possible $P(\xi) \in \mathcal{P}$.
- penalizes his choice of $P(\xi)$ by its deviation from $P$.
- the penalty function: $h(x) = x + \beta (e^x - 1)$.

- The constant $\phi > 0$ captures his degree of uncertainty aversion.
The total equity premium has three components:

1. the diffusive risk premium  
   \[ \gamma \sigma^2 \]

2. the jump risk premium  
   \[ \lambda k - \bar{\lambda} \bar{k} \]
   
   \( \bar{\lambda} = \lambda (1 + k)^{-\gamma} e^{\frac{1}{2} \gamma (1 + \gamma) \sigma_J^2}, \quad \bar{k} = (1 + k) e^{-\gamma \sigma_J^2} - 1. \)

3. the rare event premium  
   \[ \bar{\lambda} \bar{k} - \lambda^Q k^Q \]
   
   \[ \lambda^Q = \bar{\lambda} e^{a^* - \gamma b^* \sigma_J^2}, \quad k^Q = (1 + \bar{k}) e^{b^* \sigma_J^2} - 1, \]

   where \( a^* \) and \( b^* \) are the optimal parameters determined by \( \phi \).
As a rough calibration, let’s set the diffusive volatility \( \sigma = 15\% \), jump arrival intensity \( \lambda = 1/3 \), and random jump sizes with mean \( \mu_J = -1\% \) and standard deviation \( \sigma_J = 4\% \).

The observation that the total equity premium is 8% per year can be supported by various pairs of \( \gamma \) and \( \phi \):

- an investor with \( \gamma = 3.47 \) and \( \phi = 0 \).
- an investor with \( \gamma = 3.15 \) and \( \phi = 10 \).
- an investor with \( \gamma = 2.62 \) and \( \phi = 20 \).
- . . . .
For all these cases, our model implies very different compositions of the total equity premium:

<table>
<thead>
<tr>
<th>$\phi$</th>
<th>$\gamma$</th>
<th>risk premia</th>
<th>uncertainty premium</th>
<th>total equity premium</th>
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</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>diffusive</td>
<td>jump</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>3.47</td>
<td>7.80%</td>
<td>0.20%</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>3.15</td>
<td>7.09%</td>
<td>0.19%</td>
<td>0.72%</td>
</tr>
<tr>
<td>20</td>
<td>2.62</td>
<td>5.91%</td>
<td>0.15%</td>
<td>1.94%</td>
</tr>
</tbody>
</table>
Testable Implications on Options

- Unlike equity, options are non-linear in nature, with different sensitivities to diffusive shocks and jumps.
- In fact, by offering options with different degrees of moneyness and maturity, the options market provides a rich spectrum of such differential sensitivities.
- For this reason, the options market provides an ideal place for us to test our model’s prediction on rare event premia.
\[
\lambda = \frac{1}{3}, \quad \mu_J = -1\%
\]
\[
\lambda = 1/3, \quad \mu_J = -1\%
\]

\[
\lambda = 1/25, \quad \mu_J = -10\%
\]

\[
\lambda = 1/100, \quad \mu_J = -20\%
\]

\[
\lambda = 1/50, \quad \mu_J = -20\%
\]
Conclusion

- We modified the standard pure-exchange economy by adding jumps as rare events, and by allowing the representative agent to perform robust control as a precaution against possible model mis-specification with respect to rare events.

- Provided an explicitly solved equilibrium, and showed that the total equity premium has three components: the diffusive risk premium, the jump risk premium, and the rare event premium.

- Examined the testable implications of our model on the options market, and documented the importance of the uncertainty aversion toward rare events in explaining the options data.