Integrated Time-Series Analysis of Spot and Option Prices

Jun Pan
Objective

analyze the risk premia embedded in option prices

option prices ← risk premia ← dynamics ← spot prices
“Smile” Curves of S&P 500 Index Options on Nov. 2, 1993
(Source: Ait-Sahalia and Lo [1998])

Black-Scholes Implied Vol (%) vs Strike-to-Future Ratio

- 17 days
- 45 days
- 80 days
- 136 days
- 227 days
- 318 days
Related Literature

“RISK-NEUTRAL” DYNAMICS AND “SMILES”

- non-parametric: Rubinstein [1994]; Ait-Sahalia and Lo [1998].
- parametric: Bates [1997]; Bakshi, Cao, and Chen [1997].

RECONCILE SPOT AND OPTION DYNAMICS

- “peso” explanation: Aït-Sahalia, Wang, and Yared [1998].
- empirical pricing kernel and risk aversion: Rosenberg and Engle [1999]; Jackwerth [1999]; Ait-Sahalia and Lo [2000].
- market frictions: Longstaff [1995].
An Outline

- **Model:** parametric models of *risk factors* and *risk premia*.
- **Estimation:** “implied-state” generalized method of moments.
- **Empirical Results:** compelling evidence of jump-risk premia.
Risk Factors

\[
dS_t = g_t \, S_t \, dt + \sqrt{V_t} S_t \, dW_t^s + \Delta S_t - \mu S_t \, \lambda V_t \, dt \\
\]

\[
dV_t = (a + b \, V_t) \, dt + \sigma \sqrt{V_t} \, dW_t^v \\
\]

- “Brownian” shocks to \( S \) and \( V \) correlated with constant \( \rho \).
- \( \Delta S \): pure jump whose arrival intensity is \( \{ \lambda V_t, \, t \geq 0 \} \) with constant \( \lambda \). At each jump time \( T \), \( \Delta S_T / S_T \) log-normal with mean \( \mu \).
Risk Premia

• actual dynamics:

\[ dS_t = g_t \, S_t \, dt + \sqrt{V_t} \, S_t \, dW_t^s + \Delta S_t - \mu \, S_t \, \lambda \, V_t \, dt \]

\[ dV_t = (a + b \, V_t) \, dt + \sigma \sqrt{V_t} \, dW_t^v \]

• “risk-neutral” dynamics:

\[ dS_t = (r_t - q_t) \, S_t \, dt + \sqrt{V_t} \, S_t \, dW_t^{*s} + \Delta S_t - \mu^* \, S_t \, \lambda^* \, V_t \, dt \]

\[ dV_t = (a + b^* \, V_t) \, dt + \sigma \sqrt{V_t} \, dW_t^{*v} \]
Equity Risk Premia

\[ dS_t = g_t S_t \, dt + \sqrt{V_t} S_t \, dW_t^s + \Delta S_t - \mu S_t \lambda V_t \, dt \]

\[ g_t = r_t - q_t + \eta V_t + \lambda V_t (\mu - \mu^*) \]
Summary of Model Parameters

\[ \vartheta = [a, b, \sigma, b^*, \lambda, \mu, \sigma_J, \mu^*, \rho, \eta]^T. \]
Option Pricing and Option-Implied Volatility

- For an option with strike-to-spot ratio $k_t$ and maturity $\tau_t$,
  \[ C_t^\vartheta = S_t \ f(V_t, \vartheta, k_t, \tau_t), \]
  where $S_t$ and $V_t$ are the time-$t$ spot price and volatility, and $f$ is explicit up to Fourier inversion. (Heston [1993], Bates [1997], . . . .)

- **Option-Implied Volatility** $V_t^\vartheta$ defined by
  \[ C_t = S_t \ f(V_t^\vartheta, \vartheta, k_t, \tau_t). \]
  “True” volatility $V_t$ is recovered at true model parameters $\vartheta_0$:
  \[ V_t^\vartheta \bigg|_{\vartheta=\vartheta_0} = V_t. \]
Data for Model Estimation

Synchronized time-series data \( \{ S_n, C_n \} \) collected from Berkeley Options Data Base, a complete trading record of CBOE.

- \( S_n \): the date-\( n \) S&P 500 index.
- \( C_n \): the date-\( n \) price of an S&P 500 index option with
  - strike-to-spot ratio \( k_n \), and maturity \( \tau_n \).
  - on average, \( \{ k_n \} \) is 1.0002 (\( \pm \) 0.0067); \( \{ \tau_n \} \) is 31 (\( \pm \) 9) days.

Sample period extends from January 1989 to December 1996, with weekly frequency.
Estimation Strategy: “Implied-State” GMM

Option-Implied Volatility $V_{n}^{\theta}$

$C_{n} = S_{n} f(V_{n}^{\theta}, \theta, k_{n}, \tau_{n})$

Data
$\{S_{n}, C_{n}\}$

Option Pricing Model ($\varphi$)
$C_{n} = S_{n} f(V_{n}, \varphi_{0}, k_{n}, \tau_{n})$

Dynamic Model ($\varphi$)
$\{S_{n}, V_{n}\}$

Moment-Generating Function

Risk Premia

Diagnostics

Dynamics
Fundamental Moment Conditions

Let \((y_n, V_n)\) be the date-\(n\) return and volatility of the underlying.

\[
E_n(\epsilon_{n+1}) = 0, \text{ with } \epsilon_n = \begin{cases} 
y_{n+1} - E_n(y_{n+1}) \\
y^2_{n+1} - E_n(y^2_{n+1}) \\
y^3_{n+1} - E_n(y^3_{n+1}) \\
y^4_{n+1} - E_n(y^4_{n+1}) \\
V_{n+1} - E_n(V_{n+1}) \\
V^2_{n+1} - E_n(V^2_{n+1}) \\
y_{n+1}V_{n+1} - E_n(y_{n+1}V_{n+1}) \end{cases}
\]
Explicit Conditional Moments

- The joint conditional moment-generating function of $y$ and $V$

$$E_n \left[ \exp \left( u y_{n+1} + v V_{n+1} \right) \right] = \phi(u, v, V_n)$$

is known in closed form, for any $u, v \in \mathbb{R}$. (see Duffie, Pan, and Singleton [1999] for regularity conditions.)

- For any $i$ and $j$, we have

$$E_n \left( y_{n+1}^i V_{n+1}^j \right) = \frac{\partial^{(i+j)} \phi(u, v, V_n)}{\partial^i u \, \partial^j v} \bigg|_{u,v=0}.$$
“Optimal” Moment Conditions

For the chosen set of 7 fundamental moment conditions

\[ E_n(\epsilon_{n+1}) = 0, \]

the “optimal” moment conditions are

\[ E_n(Z_n \epsilon_{n+1}) = 0, \]

with “optimal” instruments (in the spirit of Hansen [1985])

\[ Z_n = \left[ E_n \left( \frac{\partial \epsilon_{n+1}}{\partial \theta} \right) \right]^\top \text{Cov}^{-1}_n(\epsilon_{n+1}). \]
An Outline of Empirical Results

SV0

- rejected with \( p \)-value < \( 10^{-10} \)
- cannot close the “gap”

SV

- rejected with \( p \)-value < \( 10^{-5} \)
- explosive “risk-neutral” volatility

SVJ0

- not rejected (\( p \)-value=0.4)
- key: jump-risk premia

with stochastic volatility

volatility risk not priced
Under the SV0 Model, $E(\epsilon_{n+1}^y) = 0$ is strongly violated!

$t$-stat: $-3.98$, rejected with $p$-value $< 10^{-4}$.

\[ \epsilon_{n+1}^y = y_{n+1}^2 - E_n(y_{n+1}^2). \]
\[ dV = (a + b^* V) \, dt + \sigma \sqrt{V} \, dW^*, \quad \hat{b}^* > 0! \]

Near-the-Money Options on 96/10/22

- SV-Model Implied
- Market Observed

Time to Expiration (month)

Black-Scholes Volatility (%)
Jump-Risk Premia *Inherently Different* from Vol-Risk Premia

![Graph showing the relationship between Black-Scholes Volatility (\%) and Time to Expiration (month)].

- **SVJ0-Model Implied**
- **SV-Model Implied**
- **Market Observed**

- Spot volatility around 11%
“Smiles” and “Smirks”
“Smiles” with the SVJ0 Model

Black-Scholes Implied Volatility (%) vs. Strike-to-Spot Ratio
Turn off the Jump-Risk Premia

Black-Scholes Implied Volatility (%) vs. Strike-to-Spot Ratio
A Sequel?

- Allow volatility to have two factors: one strongly persistent and the other quickly mean-reverting and highly volatile.

- Correlation between jump-risk premia and market volatility can be relaxed by introducing an extra factor.

- To separately identify premia for jump-timing and jump-size risks, incorporating out-of-the-money and in-the-money options will help.
Conclusions

- Jump-risk premia important not only in reconciling spot and option dynamics, but also in explaining “smiles” and “smirks.”

- Such jump-risk premia respond quickly to market volatility, becoming more prominent during volatile markets.
References


Econometric Analysis. Working Paper, J.L. Kellogg Graduate School of Management, Northwestern University.


