Integrated Time-Series Analysis of Spot and Option Prices

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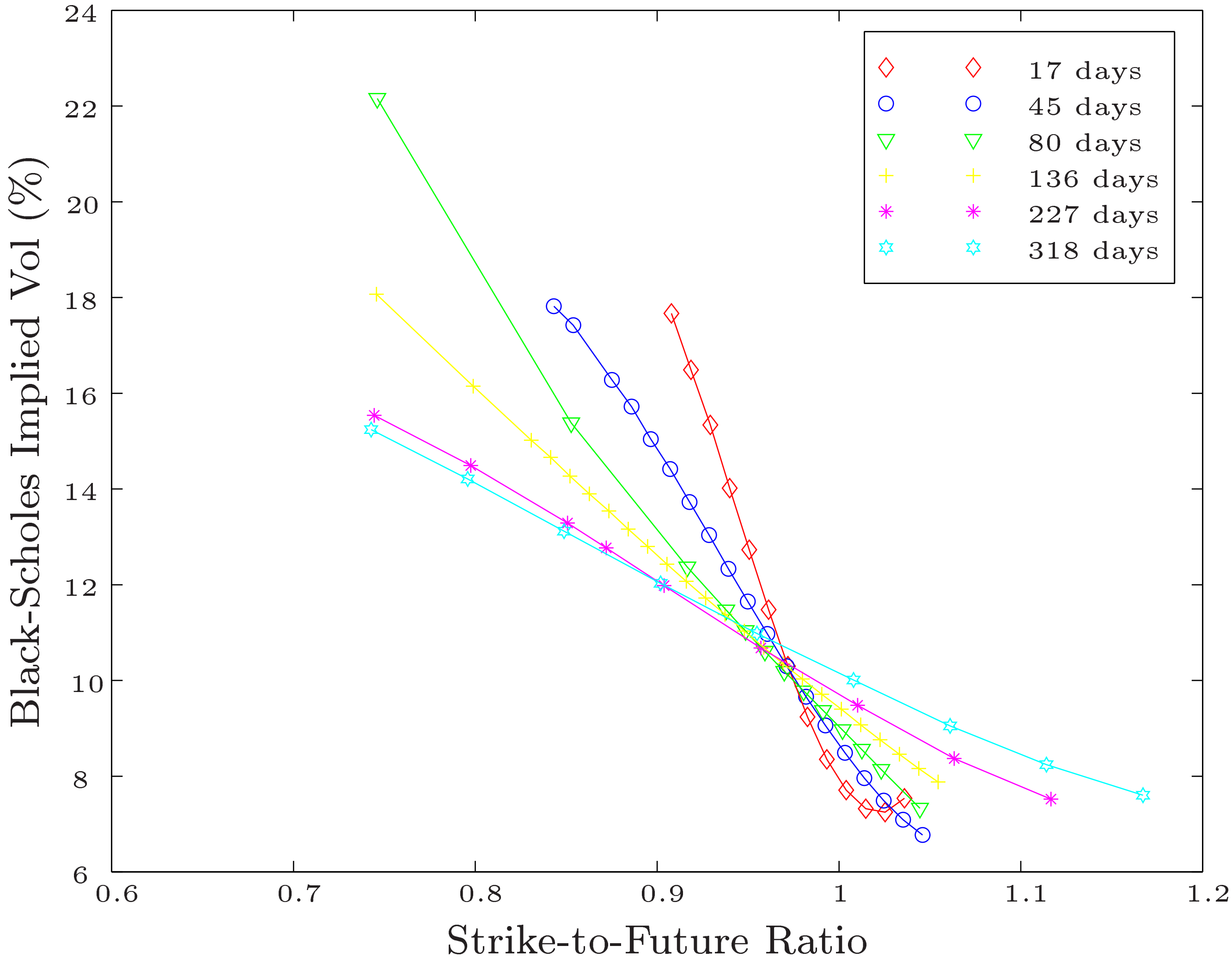
Objective

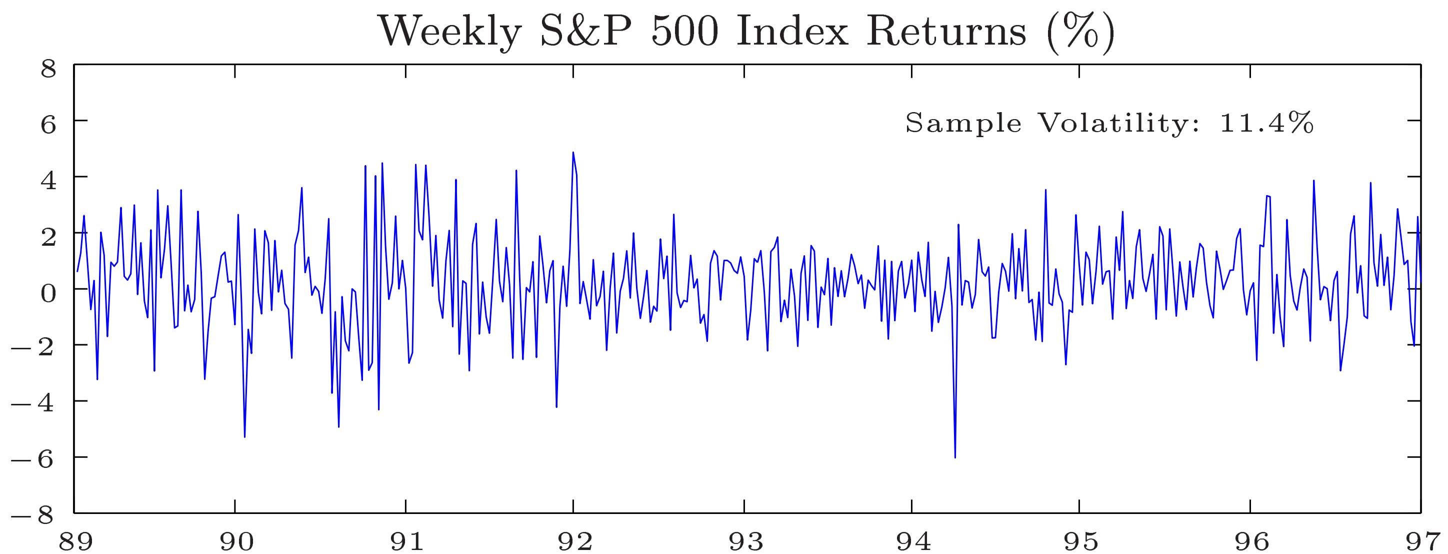
analyze the risk premia embedded in option prices

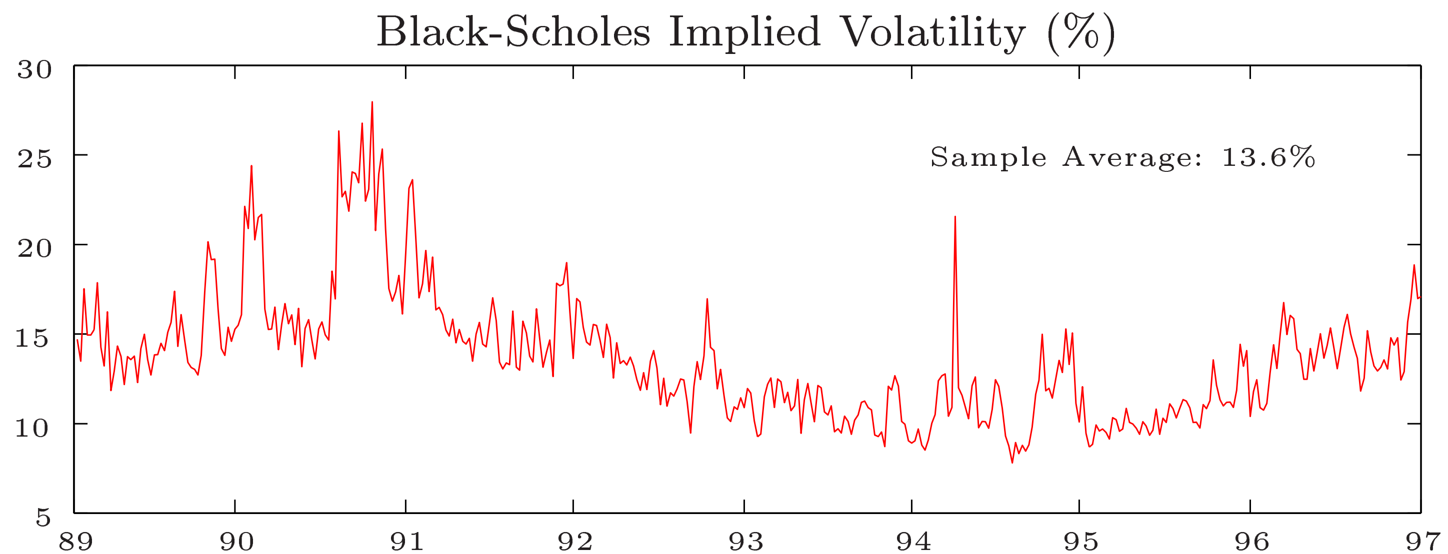
$$\begin{array}{lll} & \swarrow \ \text{risk premia} & \\ \text{option prices} & & \\ & \nwarrow \ \text{dynamics} & \longleftarrow \ \text{spot prices} \end{array}$$

"Smile" Curves of S&P 500 Index Options on Nov. 2, 1993

(Source: Ait-Sahalia and Lo [1998])




Weekly S&P 500 Index Returns (%)
Sample Volatility: 11.4%
8
6
4
2
0
−2
−4
−6
−8
89
90
91
92
93
94
95
96
97


Black-Scholes Implied Volatility (%)
Sample Average: 13.6%
30
25
20
15
10
5
89
90
91
92
93
94
95
96
97

Related Literature

"RISK-NEUTRAL" DYNAMICS AND "SMILES"

- non-parametric: Rubinstein [1994]; Ait-Sahalia and Lo [1998].
- parametric: Bates [1997]; Bakshi, Cao, and Chen [1997].

RECONCILE SPOT AND OPTION DYNAMICS

- volatility-risk premia: Guo [1998]; Benzoni [1998]; Chernov and Ghysels [1999]; Poteshman [1998].
- "peso" explanation: Aït-Sahalia, Wang, and Yared [1998].
- empirical pricing kernel and risk aversion: Rosenberg and Engle [1999]; Jackwerth [1999]; Ait-Sahalia and Lo [2000].
- market frictions: Longstaff [1995].

An Outline

- **Model:** parametric models of *risk factors* and *risk premia*.
- **Estimation:** "implied-state" generalized method of moments.
- **Empirical Results:** compelling evidence of jump-risk premia.

Risk Factors

$$dS_t = g_t\, S_t\, dt + \sqrt{V_t} S_t\, dW_t^s + \Delta S_t - \mu S_t\, \lambda V_t\, dt$$

$$dV_t = (a + b\, V_t)\; dt + \sigma \sqrt{V_t}\, dW_t^v$$

- "Brownian" shocks to S and V correlated with constant ρ.
- ΔS: pure jump whose arrival intensity is $\{\lambda V_t,\, t \geq 0\}$ with constant λ. At each jump time $\mathcal{T}$, $\Delta S_{\mathcal{T}}/S_{\mathcal{T}}$ log-normal with mean μ.

Risk Premia

- **actual dynamics:**

$$dS_t = g_t\, S_t\, dt + \sqrt{V_t} S_t\, dW_t^s + \Delta S_t - \mu\, S_t\, \lambda V_t\, dt$$

$$dV_t = (a + b\, V_t)\ dt + \sigma \sqrt{V_t}\, dW_t^v$$

- **"risk-neutral" dynamics:**

$$dS_t = (r_t - q_t)\, S_t\, dt + \sqrt{V_t} S_t\, dW_t^{*s} + \Delta S_t - \mu^*\, S_t\, \lambda^* V_t\, dt$$

$$dV_t = (a + b^*\, V_t)\ dt + \sigma \sqrt{V_t}\, dW_t^{*v}$$

Equity Risk Premia

$$dS_t = g_t\, S_t\, dt + \sqrt{V_t} S_t\, dW_t^s + \Delta S_t - \mu\, S_t\, \lambda V_t\, dt$$

$$\downarrow$$

$$g_t = r_t - q_t + \eta V_t + \lambda V_t (\mu - \mu^*)$$

Summary of Model Parameters

$$\vartheta = [\, a, b, \sigma, b^*, \lambda, \mu, \sigma_J, \mu^*, \rho, \eta \,]^\top \, .$$

Option Pricing and Option-Implied Volatility

- For an option with strike-to-spot ratio k_t and maturity τ_t,

$$C_t^{\vartheta} = S_t\, f(V_t, \vartheta, k_t, \tau_t)\,,$$

where S_t and V_t are the time-t spot price and volatility, and f is explicit up to Fourier inversion. (Heston [1993], Bates [1997], ...)

- Option-Implied Volatility V^{ϑ} defined by

$$C_t = S_t\, f(V_t^{\vartheta}, \vartheta, k_t, \tau_t)\,.$$

"True" volatility V_t is recovered at true model parameters ϑ_0 :

$$V_t^{\vartheta}\Big|_{\vartheta=\vartheta_0} = V_t\,.$$

Data for Model Estimation

Synchronized time-series data $\{S_n, C_n\}$ collected from Berkeley Options Data Base, a complete trading record of CBOE.

- S_n: the date-n S&P 500 index.
- C_n: the date-n price of an S&P 500 index option with
  - strike-to-spot ratio k_n, and maturity τ_n.
  - on average, $\{k_n\}$ is 1.0002 ($\pm$ 0.0067); $\{\tau_n\}$ is 31 ($\pm$ 9) days.

Sample period extends from January 1989 to December 1996, with weekly frequency.

Estimation Strategy: "Implied-State" GMM

Option-Implied Volatility V_n^{ϑ}

$$C_n = S_n\, f(V_n^{\vartheta}, \vartheta, k_n, \tau_n)$$

V_n^{ϑ}

Data

$\{S_n, C_n\}$

Option Pricing Model (ϑ)

$$C_n = S_n\, f(V_n, \vartheta_0, k_n, \tau_n)$$

Dynamic Model (ϑ)

$\{S_n, V_n\}$

Moment-Generating Function

Risk Premia

Diagnostics

Dynamics

Fundamental Moment Conditions

Let (y_n, V_n) be the date-n *return* and *volatility* of the underlying.

$$E_n(\epsilon_{n+1}) = 0\,, \text{ with } \epsilon_n = \begin{cases} y_{n+1} - E_n\left(y_{n+1}\right) \\ y_{n+1}^2 - E_n\left(y_{n+1}^2\right) \\ y_{n+1}^3 - E_n\left(y_{n+1}^3\right) \\ y_{n+1}^4 - E_n\left(y_{n+1}^4\right) \\ V_{n+1} - E_n\left(V_{n+1}\right) \\ V_{n+1}^2 - E_n\left(V_{n+1}^2\right) \\ y_{n+1}V_{n+1} - E_n\left(y_{n+1}V_{n+1}\right) \end{cases}$$

Explicit Conditional Moments

- The joint conditional moment-generating function of y and V

$$E_n\left[\exp\left(u\, y_{n+1} + v\, V_{n+1}\right)\right] = \phi(u, v, V_n)$$

is known in closed form, for any $u, v \in \mathbb{R}$. (see Duffie, Pan, and Singleton [1999] for regularity conditions.)

- For any i and j, we have

$$E_n\left(y_{n+1}^{i} V_{n+1}^{j}\right) = \left.\frac{\partial^{(i+j)}\phi(u, v, V_n)}{\partial^{\,i} u\, \partial^{\,j} v}\right|_{u,v=0}.$$

“Optimal” Moment Conditions

For the chosen set of 7 fundamental moment conditions

$$E_n(\epsilon_{n+1}) = 0\,,$$

the “optimal” moment conditions are

$$E_n(\mathcal{Z}_n\,\epsilon_{n+1}) = 0\,,$$

with “optimal” instruments (in the spirit of Hansen [1985])

$$\mathcal{Z}_n = \left[E_n\left(\frac{\partial \epsilon_{n+1}}{\partial \vartheta}\right)\right]^\top \mathrm{Cov}_n^{-1}(\epsilon_{n+1})\,.$$

An Outline of Empirical Results

with stochastic volatility

volatility risk not priced

SV0

- rejected with p-value $< 10^{-10}$
- cannot close the "gap"

dial in volatility-risk premia

dial in *jump* and jump-risk premia

SV

- rejected with p-value $< 10^{-5}$
- explosive "risk-neutral" volatility

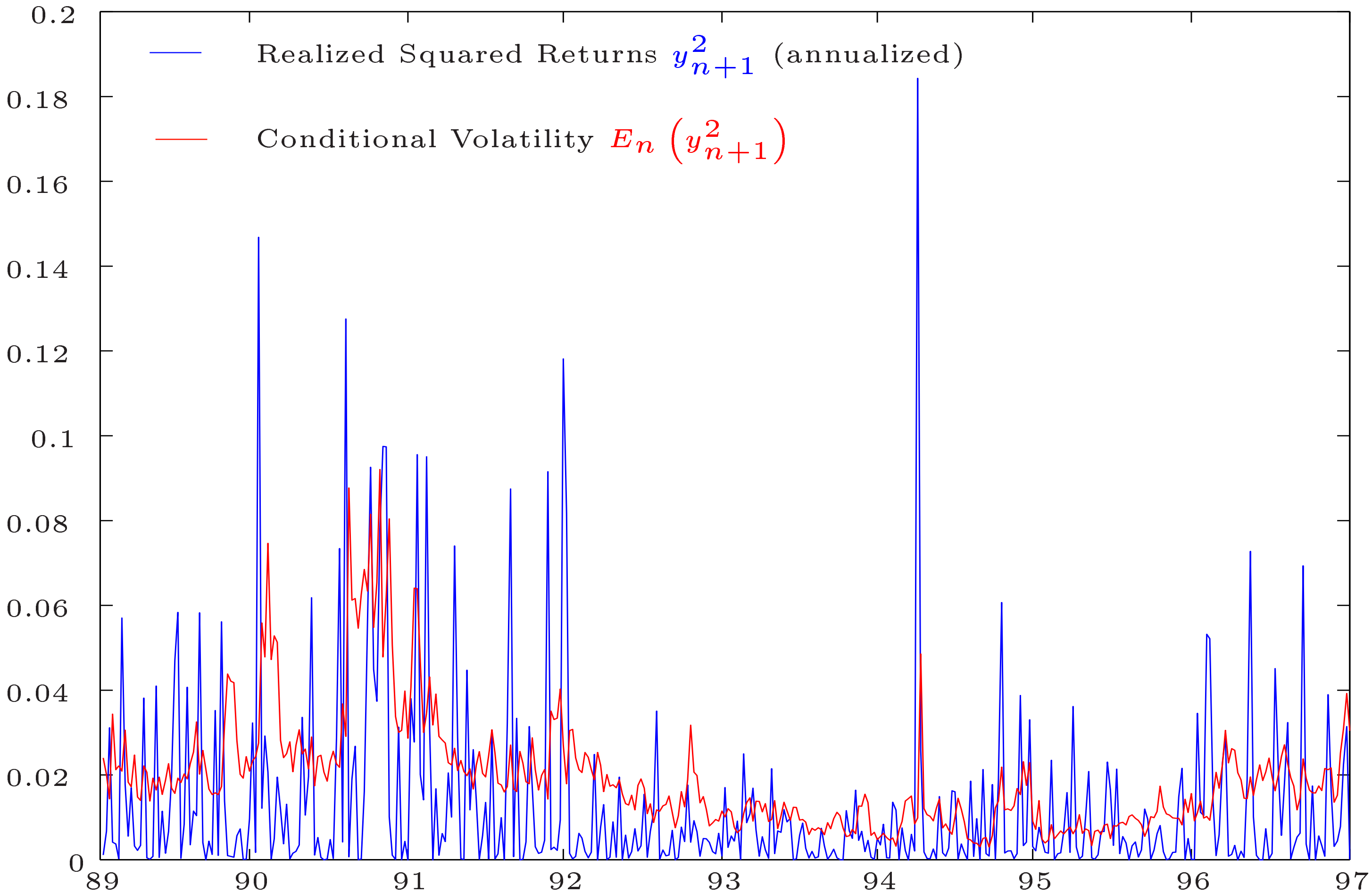
SVJ0

- not rejected (p-value=0.4)
- key: jump-risk premia

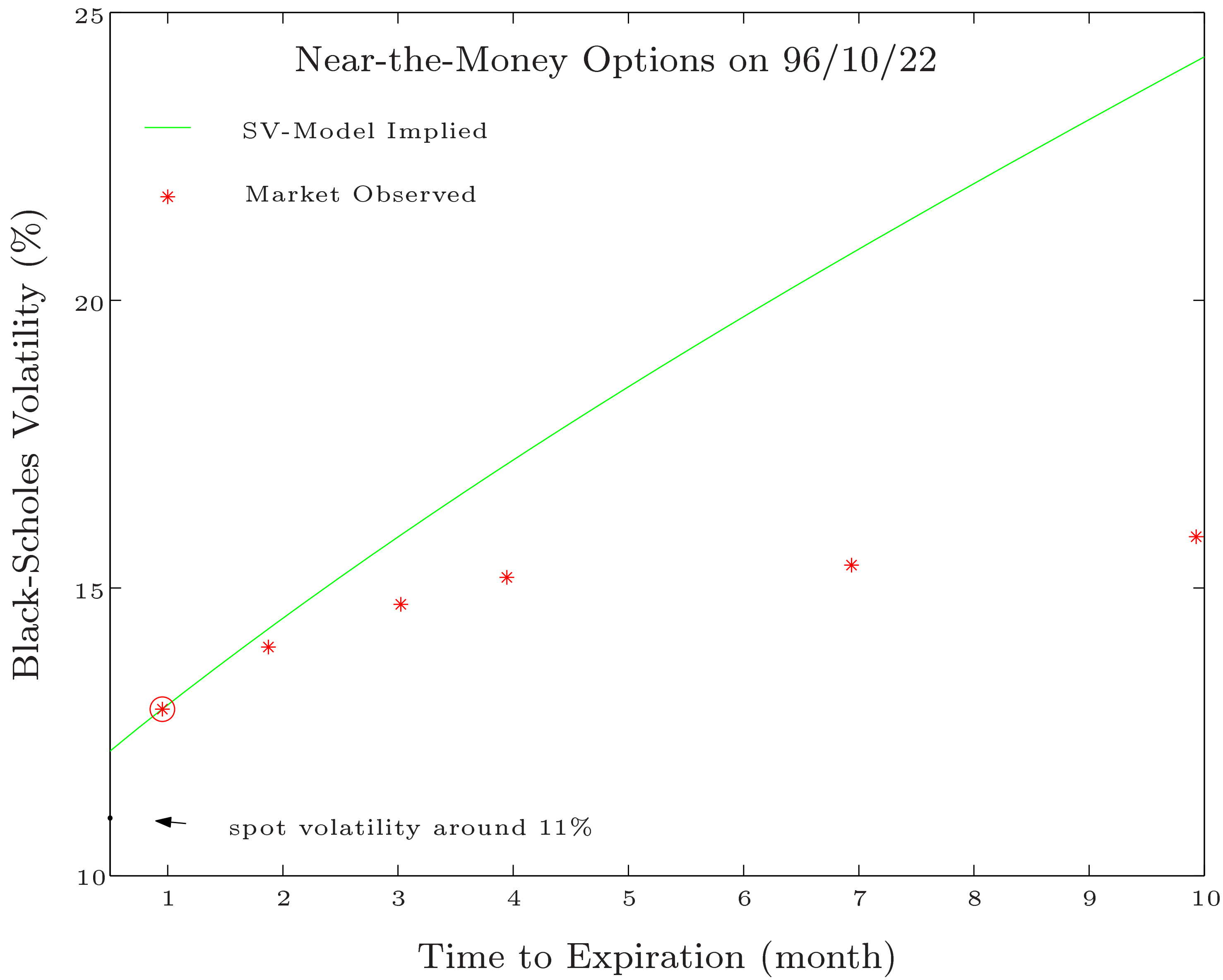
Under the SV0 Model, $E(\epsilon^{y2}_{n+1}) = 0$ is strongly violated!

t-stat: -3.98, rejected with p-value $< 10^{-4}$.

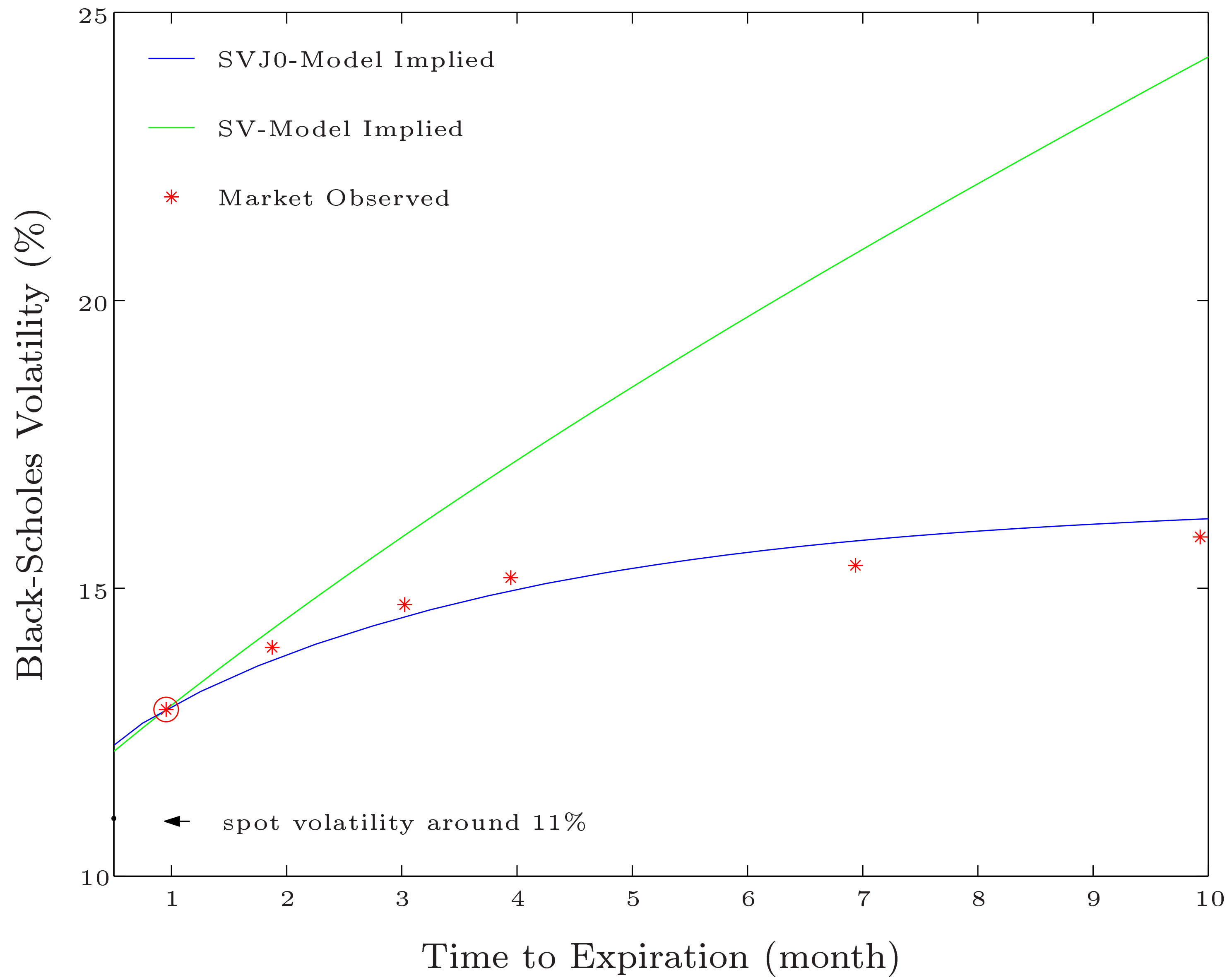
$$\epsilon^{y2}_{n+1} = y^2_{n+1} - E_n\left(y^2_{n+1}\right).$$

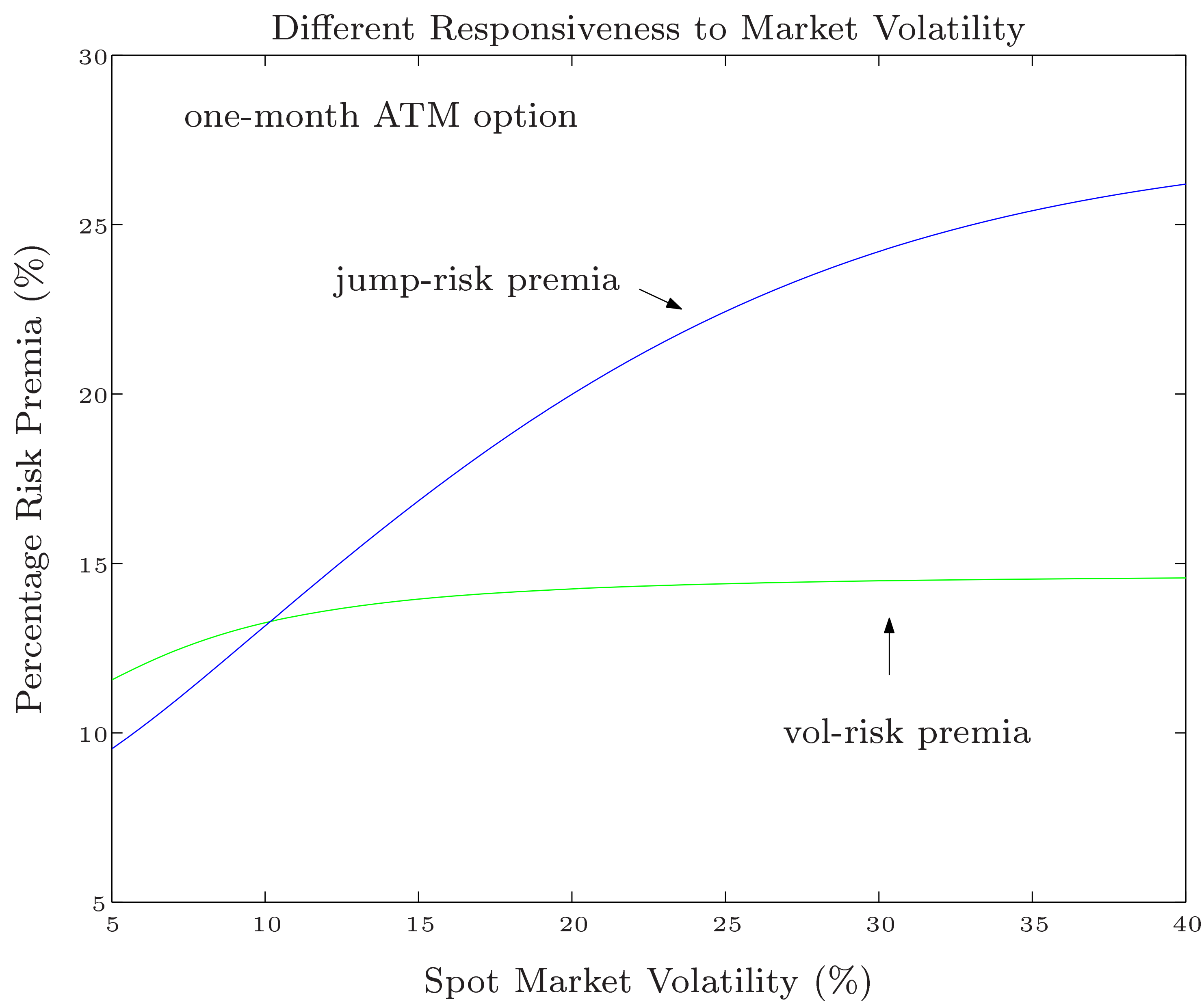


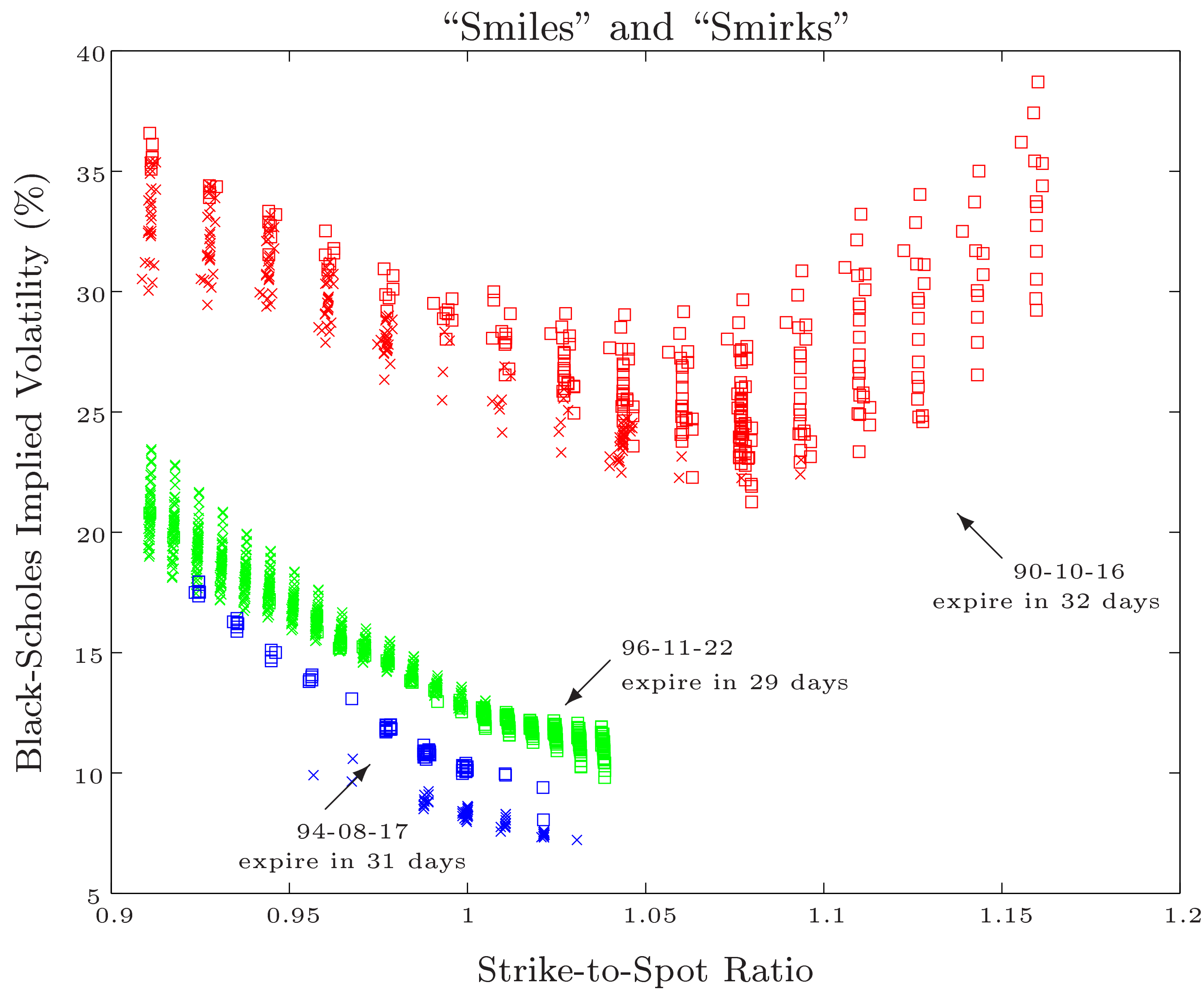
$$dV = (a + b^* V)\ dt + \sigma \sqrt{V}\, dW^*, \qquad \hat{b}^* > 0\,!$$

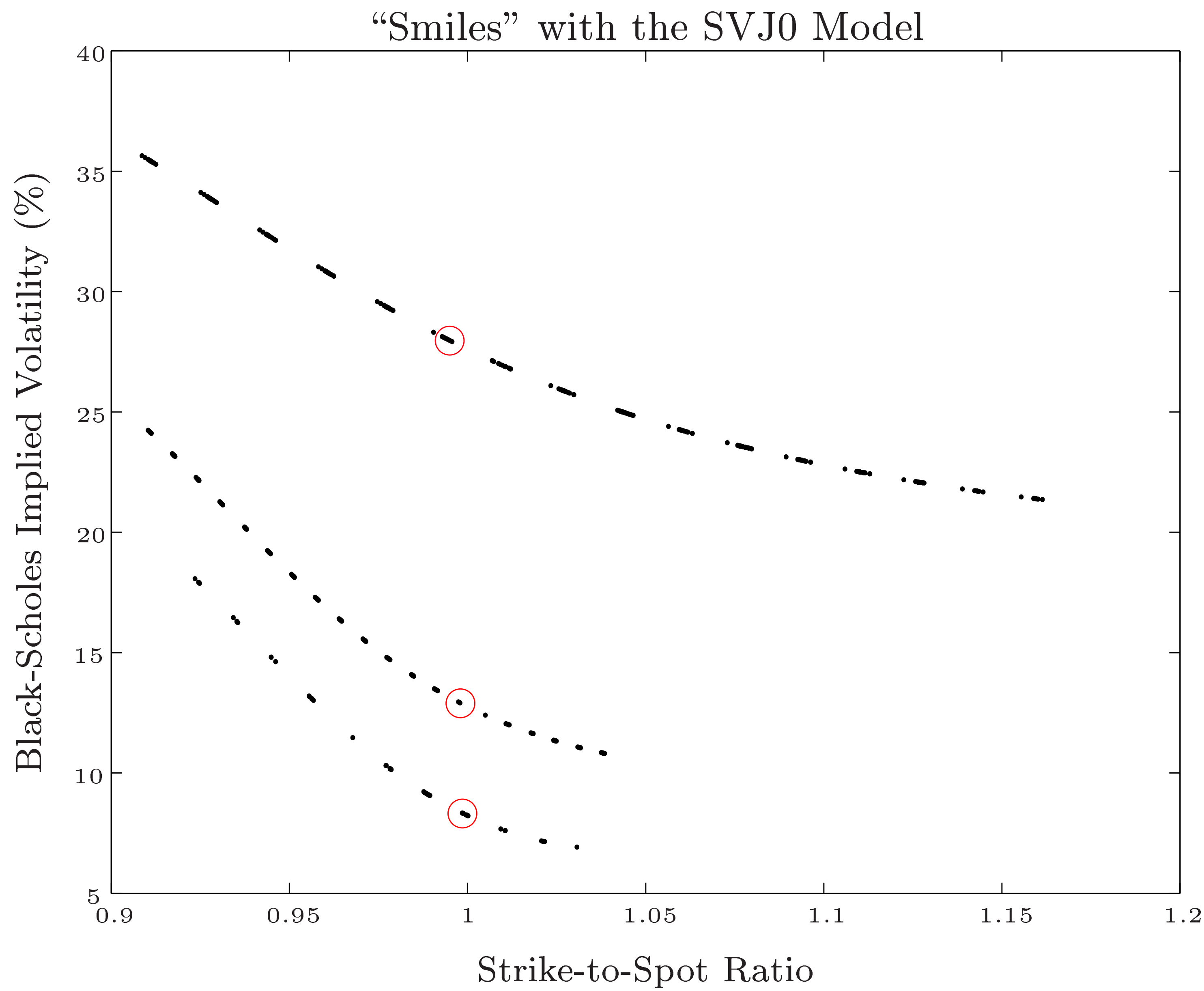


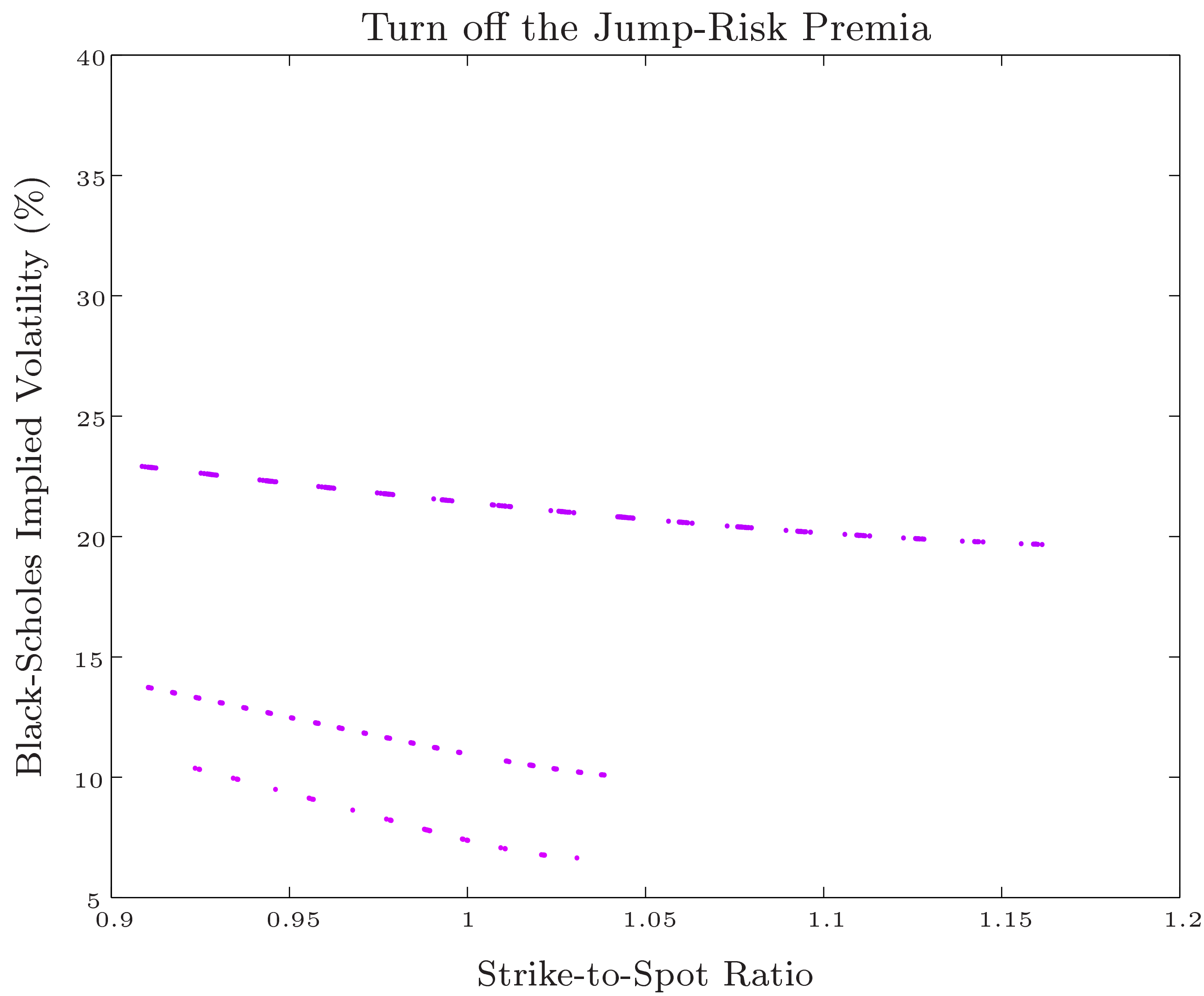
Jump-Risk Premia *Inherently Different* from Vol-Risk Premia




Different Responsiveness to Market Volatility
one-month ATM option
jump-risk premia
vol-risk premia
Percentage Risk Premia (%)
Spot Market Volatility (%)
30
25
20
15
10
5
5
10
15
20
25
30
35
40


"Smiles" and "Smirks"
Black-Scholes Implied Volatility (%)
Strike-to-Spot Ratio
40
35
30
25
20
15
10
5
0.9
0.95
1
1.05
1.1
1.15
1.2
90-10-16
expire in 32 days
96-11-22
expire in 29 days
94-08-17
expire in 31 days


"Smiles" with the SVJ0 Model
Black-Scholes Implied Volatility (%)
40
35
30
25
20
15
10
5
0.9
0.95
1
1.05
1.1
1.15
1.2
Strike-to-Spot Ratio


Turn off the Jump-Risk Premia
Black-Scholes Implied Volatility (%)
40
35
30
25
20
15
10
5
0.9
0.95
1
1.05
1.1
1.15
1.2
Strike-to-Spot Ratio

A Sequel?

- Allow volatility to have two factors: one strongly persistent and the other quickly mean-reverting and highly volatile.
- Correlation between jump-risk premia and market volatility can be relaxed by introducing an extra factor.
- To separately identify premia for jump-timing and jump-size risks, incorporating out-of-the-money and in-the-money options will help.

Conclusions

- Jump-risk premia important not only in reconciling spot and option dynamics, but also in explaining "smiles" and "smirks."
- Such jump-risk premia respond quickly to market volatility, becoming more prominent during volatile markets.

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