Transform Analysis and Asset Pricing for Affine Jump-Diffusions

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An Outline

- **Affine Jump-Diffusions**
- **Transforms**
- **Transform Inversion**
- **Some Applications**

Multi-Factor Affine Jump-Diffusions

$$dX = \mu(X)\, dt + \sigma(X)\, dW + dZ$$

- W is a standard multi-dimensional Brownian motion.
- $\mu(x)$ and $\sigma(x)\sigma(x)^\top$ are constant-plus-linear in x.
- Z is a pure jump process:
  - jumps arrive with state-dependent intensity $\lambda(X) = l_0 + l_1 \cdot X$, with constant $l = (l_0, l_1)$.
  - upon jump arrivals, random jumps with fixed probability distributions.

Characteristic Function

$$\psi(u) = E\left(e^{iu \cdot X_T}\right)$$

Moment Generating Function

$$\phi(u) = E\left(e^{u \cdot X_T}\right)$$

Bond Price

$$p(T) = E\left[\exp\left(-\int_0^T R(X_s)\,ds\right)\right]$$

Survival Probability

$$P(\tau > T) = E\left[\exp\left(-\int_0^T h(X_s)\,ds\right)\right]$$

Transform

Let X be an n-factor affine jump-diffusion, and let R be an affine discount function. let

$$\psi(u, X_0, T) = E\left[\exp\left(-\int_0^T R(X_s)\, ds\right) e^{u \cdot X_T}\right] .$$

Under technical conditions, we have

$$\psi(u, x, t) = e^{\alpha(t) + \beta(t) \cdot x} ,$$

where α and β satisfy ordinary differential equations.

Transform Inversion

Let $a, b \in \mathbb{R}^n$ and $y \in \mathbb{R}$, consider

$$G(a,b,y) = E\left[\exp\left(-\int_0^T R(X_s)\,ds\right) e^{a\cdot X_T} 1_{b\cdot X_T \le y}\right],$$

which pays $e^{a\cdot X_T}$ if $b \cdot X_T \le y$.

$$G(a,b,y) = \frac{\psi(a, X_0, T)}{2} - \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\left[\psi(a+ivb, X_0, T)e^{-ivy}\right]}{v}\,dv\,.$$

An Extension of the Option-Pricing Models of Heston [1993] and Bates [1997]

$$dS_t = (r_t - q_t)\, S_t\, dt + \sqrt{V_t} S_t\, dW_t^s + \Delta S_t - \lambda_t\, \mu_s S_t\, dt$$
$$dV_t = \kappa_v(\theta_t - V_t)\, dt + \sigma_v \sqrt{V_t} dW_t^v + \Delta V_t - \lambda_t\, \mu_v\, dt$$

- stochastic long-run mean θ : $d\theta_t = \kappa_\theta(\bar{\theta} - V_t)\, dt + \sigma_\theta \sqrt{\theta_t}\, dW_t^\theta$.
- stochastic risk-free rates r : $dr_t = \kappa_r(\bar{r} - r_t)\, dt + \sigma_r \sqrt{r_t}\, dW_t^r$.
- stochastic dividend yields q : $dq_t = \kappa_q(\bar{q} - q_t)\, dt + \sigma_q \sqrt{q_t}\, dW_t^q$.
- Double jumps: ΔS and ΔV with jump arrival intensity $\lambda_t = \bar{\lambda} + \lambda_v V_t + \lambda_\theta \theta_t + \lambda_r r_t + \lambda_q q_t$. Given jump time $\mathcal{T}$, $\Delta S_{\mathcal{T}} / S_{\mathcal{T}}$ and $\Delta V_{\mathcal{T}}$ jointly distributed with fixed probability law.

"Affinize" it!

- One can show that $X = [\ln S,\, V,\, \theta,\, r,\, q]^\top$ is affine.
- For a European-style call struck at K and expiring at T

$$C_0 = E\left[e^{-\int_0^T r_t\,dt}\,(S_T - K)^+\right]$$

$$= E\left[e^{-\int_0^T r_t\,dt}\, S_T\, 1_{S_T > K}\right] - K\, E\left[e^{-\int_0^T r_t\,dt}\, 1_{S_T > K}\right]$$

$$E\left[e^{-\int_0^T r_t\,dt}\, e^{a \cdot X_T}\, 1_{b \cdot X_T < -\ln K}\right] \qquad E\left[e^{-\int_0^T r_t\,dt}\, 1_{b \cdot X_T < -\ln K}\right]$$

where $a = [1, 0, 0, 0, 0]^\top$ and $b = -a$.

- Related Literature: Heston [1993], Bates [1997], Bakshi, Cao, and Chen [1997], Scott [1997], and Bakshi and Madan [1999]; see also Stein and Stein [1991].

Default Times $\tau_1, \ldots, \tau_n$

Suppose that there are n counterparties with default intensities $h_1(X_s), \ldots, h_n(X_s)$.

$$P\,(i \text{ is the first to default and } \tau_i = t)$$

$$= E\left[\exp\left(-\int_0^t \left[h_1(X_s) + \ldots + h_n(X_s)\right]\, ds\right) h_i(X_t)\right]\, dt$$

Extended Transform

We can further extend our transform analysis to include

$$\psi(u, X_0, T) = E\left[\exp\left(-\int_0^T R(X_s)\, ds\right) (v_0 + v_1 \cdot X_T)\, e^{u \cdot X_T}\right],$$

for some constant v_0 and v_1. Under technical conditions, we have

$$\psi(u, x, t) = \Big(A(t) + B(t) \cdot x\Big)\, e^{\alpha(t) + \beta(t) \cdot x},$$

where A, B, α and β satisfy ordinary differential equations.

Estimation of Affine Models

- BOTTOM LINE: conditional density function is not explicitly known, but both *conditional characteristic function* and *conditional moment-generating function* are.
- *characteristic function*: Singleton [1999], Jiang and Knight [1999] and Chacko and Viceira [1999].
  - *time domain:* conditional density via Fourier inversion. (*approximate likelihood:* Liu, Pan, and Pedersen [1997].)
  - *frequency domain:* empirical characteristic function.
- *moment-generating function*: Liu [1997] and Pan [1999].
  - GMM using explicit conditional moments of X.
  - optimal instruments (Hansen [1985]) can be constructed.

Conclusion

- This paper provides an analytical tool for both *asset pricing* and *model estimation* under affine jump-diffusions.
- This tool is a class of *transforms*, including Laplace and Fourier transforms.
- Probability distribution and transition density can be obtained via inversions of these transforms, including Lévy and Fourier inversions.

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