Transform Analysis and Asset Pricing for Affine Jump-Diffusions

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An Outline

- Affine Jump-Diffusions
- Transforms
- Transform Inversion
- Some Applications
Multi-Factor Affine Jump-Diffusions

\[ dX = \mu(X) \, dt + \sigma(X) \, dW + dZ \]

- \( W \) is a standard multi-dimensional Brownian motion.
- \( \mu(x) \) and \( \sigma(x)\sigma(x)^\top \) are constant-plus-linear in \( x \).
- \( Z \) is a pure jump process:
  - jumps arrive with state-dependent intensity
    \[ \lambda(X) = l_0 + l_1 \cdot X, \text{ with constant } l = (l_0, l_1). \]
  - upon jump arrivals, random jumps with fixed probability distributions.
Characteristic Function

\[ \psi(u) = E \left( e^{iu \cdot X_T} \right) \]
Moment Generating Function

\[ \phi(u) = E \left( e^{u \cdot X_T} \right) \]
Bond Price

\[ p(T) = E \left[ \exp \left( - \int_0^T R(X_s) \, ds \right) \right] \]
Survival Probability

\[ P(\tau > T) = E \left[ \exp \left( - \int_0^T h(X_s) \, ds \right) \right] \]
Let $X$ be an $n$-factor affine jump-diffusion, and let $R$ be an affine discount function. let

$$
\psi(u, X_0, T) = E \left[ \exp \left( - \int_0^T R(X_s) \, ds \right) e^{u \cdot X_T} \right].
$$

Under technical conditions, we have

$$
\psi(u, x, t) = e^{\alpha(t) + \beta(t) \cdot x},
$$

where $\alpha$ and $\beta$ satisfy ordinary differential equations.
Let $a, b \in \mathbb{R}^n$ and $y \in \mathbb{R}$, consider

$$G(a, b, y) = E \left[ \exp \left( - \int_0^T R(X_s) \, ds \right) e^{a \cdot X_T} 1_{b \cdot X_T \leq y} \right],$$

which pays $e^{a \cdot X_T}$ if $b \cdot X_T \leq y$.

$$G(a, b, y) = \frac{\psi(a, X_0, T)}{2} - \frac{1}{\pi} \int_0^{\infty} \frac{\text{Im} \left[ \psi(a + ivb, X_0, T)e^{-ivy} \right]}{v} \, dv.$$

\[ dS_t = (r_t - q_t) S_t \, dt + \sqrt{V_t} S_t \, dW^s_t + \Delta S_t - \lambda_t \mu_s S_t \, dt \]

\[ dV_t = \kappa_v (\theta_t - V_t) \, dt + \sigma_v \sqrt{V_t} dW^v_t + \Delta V_t - \lambda_t \mu_v \, dt \]

- stochastic long-run mean \( \theta \): \( d\theta_t = \kappa_\theta (\bar{\theta} - V_t) \, dt + \sigma_\theta \sqrt{\theta_t} \, dW^\theta_t \).

- stochastic risk-free rates \( r \): \( dr_t = \kappa_r (\bar{r} - r_t) \, dt + \sigma_r \sqrt{r_t} \, dW^r_t \).

- stochastic dividend yields \( q \): \( dq_t = \kappa_q (\bar{q} - q_t) \, dt + \sigma_q \sqrt{q_t} \, dW^q_t \).

- Double jumps: \( \Delta S \) and \( \Delta V \) with jump arrival intensity \( \lambda_t = \bar{\lambda} + \lambda_v V_t + \lambda_\theta \theta_t + \lambda_r r_t + \lambda_q q_t \). Given jump time \( T \), \( \Delta S_T/S_T \) and \( \Delta V_T \) jointly distributed with fixed probability law.
• One can show that $X = [\ln S, V, \theta, r, q]^\top$ is affine.

• For a European-style call struck at $K$ and expiring at $T$

$$C_0 = E \left[ e^{-\int_0^T r_t \, dt} (S_T - K)^+ \right]$$

$$= E \left[ e^{-\int_0^T r_t \, dt} S_T \mathbb{1}_{S_T > K} \right] - K E \left[ e^{-\int_0^T r_t \, dt} \mathbb{1}_{S_T > K} \right]$$

where $a = [1, 0, 0, 0, 0]^\top$ and $b = -a$.

• Related Literature: Heston [1993], Bates [1997], Bakshi, Cao, and Chen [1997], Scott [1997], and Bakshi and Madan [1999]; see also Stein and Stein [1991].
Default Times $\tau_1, \ldots, \tau_n$

Suppose that there are $n$ counterparties with default intensities $h_1(X_s), \ldots, h_n(X_s)$.

$$P(i \text{ is the first to default and } \tau_i = t)$$

$$= E \left[ \exp \left( - \int_0^t [h_1(X_s) + \ldots + h_n(X_s)] \, ds \right) h_i(X_t) \right] \, dt$$
Extended Transform

We can further extend our transform analysis to include

\[ \psi(u, X_0, T) = E \left[ \exp \left( - \int_0^T R(X_s) \, ds \right) (v_0 + v_1 \cdot X_T) e^{u \cdot X_T} \right], \]

for some constant \( v_0 \) and \( v_1 \). Under technical conditions, we have

\[ \psi(u, x, t) = \left( A(t) + B(t) \cdot x \right) e^{\alpha(t)+\beta(t)\cdot x}, \]

where \( A, B, \alpha \) and \( \beta \) satisfy ordinary differential equations.
Estimation of Affine Models

- **BOTTOM LINE:** conditional density function is not explicitly known, but both *conditional characteristic function* and *conditional moment-generating function* are.

- *characteristic function:* Singleton [1999], Jiang and Knight [1999] and Chacko and Viceira [1999].
  
  - *time domain:* conditional density via Fourier inversion. *(approximate likelihood: Liu, Pan, and Pedersen [1997].)*
  
  - *frequency domain:* empirical characteristic function.

  
  - GMM using explicit conditional moments of \( X \).
  
  - optimal instruments (Hansen [1985]) can be constructed.
Conclusion

- This paper provides an analytical tool for both asset pricing and model estimation under affine jump-diffusions.

- This tool is a class of transforms, including Laplace and Fourier transforms.

- Probability distribution and transition density can be obtained via inversions of these transforms, including Lévy and Fourier inversions.
References


Pan, J. (1999). Integrated Time-Series Analysis of Spot and Option
Prices. Working Paper, Graduate School of Business, Stanford University.

